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Quantization of constrained systems with singularities using Rieffel induction *

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Abstract

Classical constrained systems can be obtained by symplectic reduction. Many of these, including Yang–Mills fields and gravity, are singular. The presence of singularities causes great difficulties in quantizing the systems, only because the quantized Hamiltonian is not essentially self-adjoint on its natural domain. A new approach, quantization via Rieffel induction, which is known to be the quantization of classical symplectic reduction, is used. This method is explicitly applied to many singular examples studied in the literature. In each case, this new approach correctly produces a well-defined, completely specified reduced Hamiltonian and reduced state space. We then study the reduction of T^*G by the adjoint action of G (taking $G = SU(2)$, $SU(3)$ as concrete examples). This comes from Yang–Mills theory on a circle. Again, the reduced (i.e. physical) quantum Hamiltonian and quantum state space are explicitly obtained. In particular, the reduced Hamiltonian is shown to be defined by Neumann boundary conditions.

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1. Introduction

1.1. Overview

There has been a long history of studies of the quantization of constrained mechanical systems, starting with Dirac. Field theories, such as Yang–Mills and general relativity, can

* Dr. N.P. Landsman has contributed enormously to this work with his insightful criticisms.

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be elegantly formulated as infinite-dimensional constrained systems [2,15]. The so-called *symplectic reduction* procedure obtains the reduced space of physical degrees of freedom and the reduced (i.e. physical) observables from the corresponding data of the unconstrained system [1,3].

In many physically interesting cases, including gravity and Yang–Mills, the classical reduced configuration and phase space contain conical singularities [2–4,19] and boundaries. These cause great difficulties in quantization. Some work has been done on the possible effects of these singularities upon quantization [8,9,17]. In this paper, we use a new approach to the quantization of constrained systems with singularities or boundaries based on the use of Rieffel induction. We shall show that this approach is both systematic and effective in dealing with singularities; a systematic comparison with other approaches will not be attempted here, though we will comment on a case by case basis.

Rieffel induction is an operator-algebraic procedure for inducing representations from simple data. Landsman showed that Rieffel induction is the exact quantum counterpart of the classical symplectic reduction [18].

Based on this observation, our prescription for quantizing constrained systems will therefore consist of:

(1) Quantize classical data and constraints for the unconstrained classical system.
(2) Use the quantum data obtained in (1) to perform Rieffel induction, yielding quantum data for the quantum constrained system.

In the present context, classical singularities only emerge as a result of imposing constraints (which are smooth themselves) on the unconstrained classical data (which are smooth as well), hence our procedure does not 'see' the classical singularities. This does not mean that we leave the singular points out, as in some approaches [8,9,17].

While Rieffel induction quantizes general symplectic reduction, in this paper, we shall restrict ourselves to its application to Marsden–Weinstein reduction. For a detailed introduction on Marsden–Weinstein reduction consult [1].

Let (S, ω_S) be a symplectic manifold, and G a Lie group (whose Lie algebra is denoted by $\mathfrak{g}$, with dual $\mathfrak{g}^*$). Further, a smooth strongly Hamiltonian group action $\phi : G \times S \to S$ is supposed to be given so that its associated momentum map $J : S \to \mathfrak{g}^*$ is Ad^*-equivariant. There are very general conditions which ensure the existence of such momentum maps. Let $\mathcal{O}_m$ be a co-adjoint orbit containing $m \in \mathfrak{g}^*$. The choice of a co-adjoint orbit implements constraints. The constraint set is then $J^{-1}(\mathcal{O}_m)$. The reduced phase space at the given constraint level is

$$J^{-1}(\mathcal{O}_m)/G \simeq J^{-1}(m)/G_m,$$

where G_m is the stability group of m. The zero-orbit ($\{0\}$) appears as a constraint in many Hamiltonian systems, e.g. general relativity and Yang–Mills theory [2,4,5]. In fact a 'shifting trick' can formally make any constraint appear as a zero-level constraint. We therefore will mainly study the zero-level case. If $S = T^*Q$ and the action of G on S is lifted from some action on Q, then we have

$$J^{-1}(0)/G \simeq T^*(Q/G),$$

given that the quotient manifold Q/G is smooth.

When the group action and the constraint level satisfy certain conditions, $J^{-1}(\mathcal{O})/\mathcal{G}$ will be a smooth manifold with a symplectic form induced from the original one by the reduction process. Unfortunately, such 'ideal' conditions do not hold in various physically interesting cases. Singularities can occur in both the constraint set $J^{-1}(\mathcal{O})$ itself, or as a result of the quotient by G. In both cases singularities occur at points with higher symmetries (than neighbouring points), i.e. their stability groups have higher dimensions. These points are frequently those that are physically interesting. In fact these are often the (relative) equilibria of the original system [19,20]. Detailed discussions on these issues can be found in [3,19,29].

The reduced manifold is actually quite well behaved even if it is singular. In [29], it is proved that provided the group action is proper, the reduced space $J^{-1}(\mathcal{O})/\mathcal{G}$ is always stratified by symplectic manifolds. For the present work we just need to know that if S is connected, then there is a unique connected open dense subset of $J^{-1}(\mathcal{O})/G$ (termed *maximal stratum*) that is a smooth symplectic manifold itself, i.e. the singular reduced space is basically a 'good' symplectic manifold with 'lower-dimensional strata' (boundaries and singularities) attached to it.

Hence, when Q/G is singular, the relation $J^{-1}(0)/G = T^*(Q/G)$ continues to be meaningful for an open dense subset of Q and an open dense subset of $J^{-1}(0)/G$, both of which are smooth manifolds in the present setup. (From here on, if the reduced manifold Q/G is singular then any mention of the above relation will be in this sense.)

The quantized versions of classical systems of this kind are highly relevant: most finite-dimensional mechanical systems fall into this category; gravity and Yang–Mills fields are infinite-dimensional versions of the same situation [2,6,15,21].

In essence, the standard strategy in quantizing systems of this kind is to quantize the reduced classical system on the maximal stratum directly, and then study the resulting quantum system [9]. As a result of leaving out the singularities, the naively quantized Hamiltonians are usually not essentially self-adjoint (e.s-a.) on their natural domain. Therefore, it is necessary to study the family of self-adjoint (s-a.) extensions. Each extension corresponds to a different quantum system and each defines, for example, a different scattering theory. In this approach, there is no good theoretical reason to exclude any of them and yet only one is physically correct.

By solving certain concrete problems, we demonstrate that the quantization scheme based on Rieffel induction does not have these defects. It is more effective in handling situations with singularities. Furthermore, the procedure is mathematically well defined.

In Section 1.2, we shall introduce Rieffel induction in the present context and develop its properties.

In Section 2, we shall carry out more explicit reductions in the context of proper and isometric group actions. The results will be directly applicable to a wide class of problems, including the examples treated in the subsequent sections.

In Section 3, we shall apply this technique to a selection of classical singular reduction problems studied in [3,19]. We include the standard reduction of $T^*\mathbb{R}^n$ by $SO(n)$, the most straightforward classical system containing singularities after reduction. There will be an example of Sniatycki and Weinstein [30] to demonstrate the striking difference between our method and theirs.

In Section 4, we study dynamics on the quotient manifolds $G/Ad\text{-}G$, where G is a compact, semi-simple, simply connected Lie group. The cases of $SU(2)/Ad\text{-}SU(2)$ and $SU(3)/Ad\text{-}SU(3)$ are explicitly treated for concreteness. This is motivated by the fact that the dynamics of 2-D Yang–Mills fields on a circle with structure group G reduces to dynamics on $G/Ad\text{-}G$ [6,21]. The problem has been treated by many authors. In one way or another, the physical quantum Hamiltonian has been formally obtained. However, the precise domain of the reduced Hamiltonian has not been derived. It is difficult to obtain because of the singularities in $G/Ad\text{-}G$. Together with another paper [31], we show that Rieffel induction provides an attractive, mathematically rigorous framework for the quantization of such field theories, where the above issue and many others (e.g. the precise analytic structure of the group of gauge transformations) are cleanly dealt with. As a result, a mathematically well-defined, completely specified quantized theory is obtained. In this paper, we concentrate on the issue of singularities and the reduced (i.e. physical) Hamiltonian.

1.2. Quantization with Rieffel induction

First let us set up the background, from which this quantization procedure emerges. Let the classical configuration space be a smooth manifold Q with an action of a Lie group G. The phase space $S = T^*Q$ carries the lifted action ϕ of G, with equivariant momentum map $J : T^*Q \mapsto \mathfrak{g}^*$. The constraint is in the form of $J = m, m \in \mathfrak{g}^*$. The reduced phase space is $J^{-1}(\mathcal{O}_m)/G$, where $\mathcal{O}_m$ is the co-adjoint orbit containing m.

We can quantize the unconstrained classical data in the standard fashion. The quantized unconstrained phase space is $\mathcal{H} = L^2(Q, \mathrm{d}\mu)$, [1]. For simplicity, we assume the existence of a G-invariant measure μ on Q. If $\psi \in \mathcal{H}$ and ϕ_a is the transformation corresponding to $a \in G$ under action ϕ, the representation $U(a)\psi = \psi \circ \phi_{a^{-1}}$ quantizes the classical group action. When the classical group action on the phase space S is not lifted from Q, the representation U is given by $\mathrm{e}^{\mathrm{i}\xi\cdot\hat{J}}$ where $\hat{J}$ is the quantized momentum map, and ξ are coordinates in the Lie algebra [12].

To understand the quantization of the constraints we have to bring in the broader perspective. The classical constraint algebra is the Poisson algebra $C^\infty(\mathfrak{g}^*)$. Its quantization is $C^*(G)$ [28], the convolution algebra of compactly supported continuous functions on G, completed in the C^*-algebra norm. Classically the inclusion map i of the orbit $\mathcal{O}_m$ into $\mathfrak{g}^*$ induces, by pull-back i^*, a representation of the Poisson algebra $C^\infty(\mathfrak{g}^*)$ in $C^\infty(\mathcal{O}_m)$. This is quantized by the situation in which the quantum constraint algebra $C^*(G)$ is represented in the C^*-algebra of bounded linear operators on a Hilbert space. Such representations π are in 1–1 correspondence with the unitary representations U of the group G on that Hilbert space [24]. It is in this sense that we say that the classical constraint $\mathcal{O}_m \hookrightarrow \mathfrak{g}^*$ is quantized by an irreducible unitary representation $U_\chi(G)$ on a Hilbert space $\mathcal{H}_\chi$. Further, the momentum map $J : S \to g^*$ is seen as a classical representation J^* of the constraint algebra $C^\infty(g^*)$. This is similarly quantized by the unitary representation U of G on $\mathcal{H}$, regarded as a representation of $C^*(G)$. To summarize, we have

$$C^\infty(S) \overset{J^*}{\leftarrow} C^\infty(\mathfrak{g}^*) \overset{i^*}{\rightarrow} C^\infty(\mathcal{O}_m), \qquad \mathcal{L}(\mathcal{H}) \overset{\pi}{\leftarrow} C^*(G) \overset{\pi_\chi}{\rightarrow} \mathcal{L}(\mathcal{H}_\chi),$$

where $\mathcal{L}(\cdot)$ denotes the space of linear operators on a Hilbert space. Classically, the so-called weak observable algebra is the Poisson commutant of the sub-algebra formed by the image of the representation J^* of the constraint algebra in $C^\infty(S)$, i.e. the space of functions invariant under the action of the group on S. Likewise, the weak quantum observables are represented by the commutant of the operator-algebra representing $C^*(G)$ on $\mathcal{H}$. Equivalently, the operators in the weak observable algebra commute with any operator of the form $U(a)$, $a \in G$. (All of the above can be found in [18].)

The above quantized data for the unconstrained system is precisely the input data of a construction called Rieffel induction, discussed in [10,27]. We will just recall its basic constructions in the present context.

Typically, the input data consist of two C^*-algebras $\mathcal{A}$, $\mathcal{B}$ with left, right representations, respectively, in the space of linear operators $\mathcal{L}(L)$ on L, where L is a linear space. In physics, L is a (pre) Hilbert space. Its completion is the quantized unconstrained phase space.

The algebra $\mathcal{A}$ is to be the quantization of the classical weak observable algebra. It can also be an algebra of unbounded operators representing observables, which needs more care with the domain of definition of the objects involved. We shall state the Rieffel induction procedure for bounded observables for the moment, and the slight difference in dealing with unbounded observables is mentioned afterwards.

The algebra $\mathcal{B}$ is the quantization of the constraint algebra, i.e. $C^*(G)$. It has a representation π_χ on a certain Hilbert space $\mathcal{H}_\chi$. This representation π_χ corresponds to an irreducible unitary representation U_χ of the group G on $\mathcal{H}_\chi$. The aim is to construct a representation for $\mathcal{A}$ on some new Hilbert space $\mathcal{H}^\chi$, which turns out to be the quantized reduced phase space.

The key is a sesquilinear form, which takes values in $\mathcal{B} = C^*(G)$, called rigging map $\langle\ ,\ \rangle_\mathcal{B}$ from $L \times L$ to $\mathcal{B}$, satisfying the conditions defining sesquilinear forms, as well as

$$\langle\phi, \psi B\rangle_\mathcal{B} = \langle\phi, \psi\rangle_\mathcal{B} B \quad \text{and} \quad \langle A\phi, \psi\rangle_\mathcal{B} = \langle\phi, A^*\psi\rangle_\mathcal{B} \tag{1.1}$$

for all $\phi, \psi \in L$ and $A \in \mathcal{A}$, $B \in \mathcal{B}$.

If $\pi_\chi(\langle\phi\ \phi\rangle_\mathcal{B}) \geq 0$, $\forall\phi \in L$ then we do the following:

(1) $L \otimes \mathcal{H}_\chi$ with $(\ ,\ )_0$ s.t.

$$(\phi \otimes v, \psi \otimes w)_0 = (\pi_\chi(\langle\psi, \phi\rangle_\mathcal{B})v, w)_{\mathcal{H}_\chi}. \tag{1.2}$$

This is positive definite [18].

(2) Quotient $L \otimes \mathcal{H}_\chi$ with the kernel $\mathcal{N}_0$ of $(\ ,\ )_0$, and complete it in the norm derived from $(\ ,\ )_0$.

(3) The completion gives a Hilbert space $\mathcal{H}^\chi$, carrying a representation π^χ of any weak observable $A \in \mathcal{A}$, given by the representation defined by

$$\pi^\chi(A)[\psi \otimes u] = [A\psi \otimes u], \tag{1.3}$$

where $\psi \otimes u \in L \otimes \mathcal{H}_\chi$, and $[\cdots] : L \otimes \mathcal{H}_\chi \mapsto \mathcal{L} \otimes \mathcal{H}_\chi/\mathcal{N}_I$. This representation is well-defined.

Without mentioning any subtleties, we apply the above to the problem in hand. Let dg be an invariant measure of G, for convenience (if assuming otherwise, then insert modular function appropriately). Then the sesquilinear form required is given in [18]. In our case, we have

$$\pi_\chi(\langle\psi_1, \psi_2\rangle_{\mathcal{B}}) = \int_G \mathrm{d}g (U(g)\psi_2, \psi_1)_{\mathcal{H}} U_\chi(g). \tag{1.4}$$

We find a suitable subspace $L \overset{\text{dense}}{\subset} \mathcal{H}$, on which the above is well-defined. Then form the space $L \otimes \mathcal{H}_\chi$. (In case of compact G, L can be taken as $\mathcal{H}$.)

Given $\psi_1 \otimes u_1, \psi_2 \otimes u_2 \in L \otimes \mathcal{H}_\chi$, by substituting (1.4) into (1.2), we have

$$(\psi_1 \otimes u_1, \psi_2 \otimes u_2)_0 = \left(\int_G \mathrm{d}g (U(g)\psi_1, \psi_2)_{\mathcal{H}} U_\chi(g) u_1, u_2 \right)_{\mathcal{H}_\chi}. \tag{1.5}$$

Consider the example where G is compact. We have

$$\int_G \mathrm{d}g (U(g)\psi_1, \psi_2)_{\mathcal{H}} (U_\chi(g)u_1, u_2)_{\mathcal{H}_\chi}$$
$$= \int_G \mathrm{d}g (U \otimes U_\chi(g))\psi_1 \otimes u_1, \psi_2 \otimes u_2)_{\mathcal{H}\otimes\mathcal{H}_\chi}. \tag{1.6}$$

Remark. If the constraint is at the zero level, then $\mathcal{H}_\chi$ is $\mathbb{C}$ and U_χ is trivial. In that case the inner product in the above integral reduces to the one on $\mathcal{H}$ alone with u's left out. On the other hand, for general constraints we could always re-define $U \otimes U_\chi$ as U' and $\mathcal{H} \otimes \mathcal{H}_\chi$ as $\mathcal{H}'$. Hence any constraint can be formally shifted to a zero-level constraint. This is a remarkable quantum analogue of the classical shifting trick.

Let P_{Id} be the orthogonal projection onto the subspace of $\mathcal{H}\otimes\mathcal{H}_\chi$ that transforms trivially under $U \otimes U_\chi$. By a well-known property of compact groups, (1.6) can be expressed in terms of P_{Id}. For $\Psi \in \mathcal{H} \otimes \mathcal{H}_\chi$, we have

$$(\Psi_1, \Psi_2)_0 = (P_{\mathrm{Id}}\Psi_1, P_{\mathrm{Id}}\Psi_2)_{\mathcal{H}\otimes\mathcal{H}_\chi}. \tag{1.7}$$

Immediately, we have

$$(L \otimes \mathcal{H}_\chi)/\mathcal{N}_0 \simeq P_{\mathrm{Id}}(L \otimes \mathcal{H}_\chi). \tag{1.8}$$

The completion of $L \otimes \mathcal{H}_\chi/\mathcal{N}_0$ in $(\ ,\ )_0$ gives $\mathcal{H}^\chi = P_{\mathrm{Id}}(\mathcal{H} \otimes \mathcal{H}_\chi)$. Note that had we chosen $L = \mathcal{H}$, we would have obtained the same result. This choice is only possible for compact groups. In general (precisely when Dirac's constrained quantization formalism breaks down), the linear space L often needs to be carefully chosen (see [18] and Section 3.1 for this point).

The reduced quantum observable, i.e. the representation $\pi^\chi(T) = T^\chi$ of a weak observable T on $\mathcal{H}^\chi$, is given by (1.3). Generally, in the case when T comes from a weak observable algebra and T^χ is bounded, the domain problem for T^χ is trivial, and T^χ is guaranteed to be s-a.

We are mainly interested in the case where the observable T is an unbounded operator with domain $D(T)$. Then the domain of the reduced observable is also defined by (1.3), namely the largest subspace of $\mathcal{H}^{\chi}$ on which (1.3) is well-defined. It is easy to see that the reduced observables are at least always symmetric, cf. (1.1). We emphasize that in the Rieffel induction approach, the domain of any reduced observable is rigorously defined, and that its self-adjointness is a very tractable and well-defined mathematical problem.

In the above compact group case, since P_{Id} and observables commute, the reduced observable is simply the restriction of the representation $T \otimes \mathbb{I}_{\chi}$ of T on $\mathcal{H} \otimes \mathcal{H}_{\chi}$, to the subspace $\mathcal{H}^{\chi}$. That is

$$T^{\chi} = (T \otimes \mathbb{I}_{\chi})|_{P_{\mathrm{Id}}(\mathcal{H}\otimes\mathcal{H}_{\chi})}. \tag{1.9}$$

In this case the domain of the reduced observable is simply

$$P_{\mathrm{Id}}(D(T) \otimes \mathcal{H}_{\chi}). \tag{1.10}$$

Self-adjointness or essential self-adjointness on the above domain can be proved directly as a simple exercise for the reader. It also follows from a theorem of Nussbaum [22] on self-adjointness, which can be generalized to include essential self-adjointness (the proof can be obtained from the author on request).

When G is not compact, all of the above still applies. The only difference is that $P_{\mathrm{Id}}\mathcal{H}$, which is still the Hilbert space of G-invariant states, is now no longer a subspace of $\mathcal{H}$, cf. Sections 2.4 and 2.5. The only crucial requirement for Rieffel induction to work is that the symmetry group should have a measure with certain properties. [2]

Note, then, that so long as the observables are given for the quantized unconstrained system, Rieffel induction will guarantee that the reduced observables are also well-defined and completely specified, whether or not the reduced classical system contains singular points. In what follows, we say 's-a. domain' for 'domain of self-adjointness', and, similarly, 'e.s-a. domain'.

2. Radial part of an operator and Rieffel induction

2.1. *Some basic geometric concepts*

In a large class of problems that we shall encounter, the classical configuration space Q is a Riemannian manifold whose metric g is invariant under the symmetry group G acting by diffeomorphism on Q. By a theorem of Palais, the action is therefore proper [23]. For convenience we assume that the reduced space $Q_0 = Q/G$ is connected. In addition, we assume that the group G is locally compact and unimodular. The unimodularity assumption is made for convenience, cf. [18]. Without this assumption, we merely need to insert the

[2] We have demonstrated that even the infinite-dimensional groups of gauge transformations have the required properties [31].

modular function into appropriate places (e.g. into the definition of the unitary representation of G).

The classical unconstrained free Hamiltonian is defined by the metric

$$H(p, q) = g'(q)(p, p), \tag{2.1}$$

where $q \in Q$, and g' is the inverse metric, acting on the momentum vector p.

Therefore, by a well-known quantization procedure (see [1]), the unconstrained free quantum Hamiltonian is the negative of the Laplace–Beltrami operator, denoted $-\Delta$, of Q. In coordinate form, it looks like

$$\Delta = \frac{1}{\sqrt{\det(g)}}\partial_k \circ g^{ik}\sqrt{\det(g)}\partial_i, \tag{2.2}$$

where g^{ik} is the inverse metric in coordinate form.

It is e.s-a. on $C_c^\infty(Q)$ if the classical Hamiltonian flow is defined for all times (i.e. geodesically complete) [1]. We shall assume this to be the case.

In this section we shall study what is termed as the 'radial part' of Δ. The techniques developed here will be essentially a generalization of Helgason's treatment on invariant operators on smooth manifolds in [14]. We shall see that the results will be highly relevant to the quantization of classical reduction at zero level of the momentum map.

Helgason defined the radial part of any differential operator D as follows:

Given some manifold M with a proper, isometric group action, we first define a transverse sub-manifold W of M as some sub-manifold satisfying

(a) $G \cdot w \cap W = \{w\}$,

(b) $T_w M = T_v W \oplus T_w(G \cdot w)$ (2.3)

for all $w \in W$. Then the radial part D^0 of any D is defined as

$$D^0 f|_W = (Df)|_W$$

for all locally G-invariant, smooth functions f on M.

By definition, D^0 is defined on the set of locally invariant functions on M restricted to W. If W itself is a smooth sub-manifold, the set of functions obtained by restriction is the set of smooth functions on W.

We shall extend these concepts to cases where M and W may only be the *closure of some manifolds* embedded in some larger space.

Under the assumption of proper and isometric action, the isotropy group of any $q \in Q$ must be compact [9], and the orbit $G \cdot q$ can be identified with (i.e. diffeomorphic to) the homogeneous space G/G_q [13]. In our case, the space G/G_q is always some smooth manifold. There is an equivariant diffeomorphism between G/G_q and $G/G_{q'}$, if and only if G_q and $G_{q'}$ are conjugate to each other [7,13].

An orbit is said to have type $[H]$, where $[H]$ is the conjugacy class of a closed subgroup H of G, if the isotropy group of any points on the orbit belongs to the conjugacy class of H. An orbit through point q of Q is called *maximal* if the isotropy group G_q is conjugate to a subgroup of every other isotropy group associated with the given action. If an orbit has dimension less than that of maximal orbits, it is said to be *singular* [7]. Apart from the two types, there is also the so-called *exceptional* type, meaning that the orbit is not maximal but has the same dimension of the maximal orbits [7]. Since they will have the same treatment as singular types, we shall generally use the term *non-maximal* to describe all points having non-maximal orbit types, and use a tilde to indicate these points. We abuse the terminology a little to refer to the *type* of a point in Q and the type of an isotropy group. Note that an isotropy group of *maximal type* in fact is in a sense the smallest.

Define $Q^{(H)} = \{q \in Q \mid G_q \text{ is of type } [H]\}$. Denote $Q^{(H)}/G$ as $Q_0^{(H)}$. The sets just defined can be analysed on the basis of the concept of a *slice*, cf. Appendix A.1. For proper, isometric group actions, at every point q of Q, there exists a sub-manifold $S_q \subset Q$, a slice containing q [4].

The existence of slices implies that each stratum $Q^{(H)}$ is a smooth (Riemannian) manifold [23]. Further, the stratum $Q^{(H)}$ that corresponds to points of maximal orbits, denoted as Q^*, is dense in Q, cf. [9]. Let $Q_0^* = Q^*/G$ denote the set of maximal orbits. The set $Q_0 - Q_0^*$ contains the non-maximal orbits. The reduced space $Q_0^{(H)}$, in particular Q_0^*, is a Riemannian manifold with a metric reduced from the metric on Q. Hence we see that Q_0 is 'pieced' together by smooth (Riemannian) manifolds. These pieces are called *strata*. The maximum stratum Q_0^* is connected, cf. [9]. The quotient map $\pi : Q \mapsto Q/G$ is smooth when restricted to each of the stratum $Q^{(H)}$. Consequently, a metric is induced for each stratum of the quotient space Q_0. It is tempting to 'extend' this to a metric defined for Q_0 as a whole. The problem is that the tangent bundle over Q_0 is yet to be defined. Fortunately, the Riemannian manifold Q_0^* is dense in the reduced space Q_0 by the definition of quotient topology. This enables us to define some kind of tangent bundle over Q_0.

First, we can define a smooth structure on Q_0. There are essentially two approaches [3,29]. The first one is to define the smooth structure via the quotient map π,

$$C^\infty(Q_0)_{(0)} = \{f : Q_0 \mapsto \mathbb{R}: f \circ \pi \in C^\infty(Q)\}. \tag{2.4}$$

Let us denote this structure as $C^\infty(Q_0)_{(0)}$. This set of functions will play an important role later. The smooth structure $C^\infty(Q_0)_{(0)}$ is isomorphic to the vector space $C^\infty(Q)^G$ of G-invariant smooth functions.

The second definition of smooth structure (Whitney-smooth functions) may be used when the quotient space is embedded in some smooth manifold, in which case the smooth functions on Q_0 are simply obtained from restriction of smooth functions on the embedding manifold. We denote this structure as $C^\infty(Q_0)$. This second definition may sound limited, but in [29] it was shown that a quotient space, such as the one we are interested in, can always be embedded into some Euclidean space.

In defining a smooth structure on Q_0, we have implicitly defined tangent bundles of all orders for Q_0. On the dense subset Q_0^*, which is already a smooth manifold, the two smooth structures give the same tangent bundles (of all orders). Note that the Whitney-smooth structure contains $C^\infty(Q_0)_{(0)}$, since evidently $C^\infty(Q_0)_{(0)}|_{Q_0^*} \subseteq C^\infty(Q_0^*) = C^\infty(Q_0)|_{Q_0^*}$.

From now on, we use the following definition for a smooth map on the reduced space Q_0:

Definition. Any map τ defined on Q_0 into Q is called *Whitney-smooth* if $\forall f \in C^\infty(Q)$, $f \circ \tau \in C^\infty(Q_0)$.

This is justified since such a map will be infinitely differentiable with respect to the tangent bundles defined below by Whitney-smooth structure.

We now take a closer look at how the tangent bundles may be defined. Let us first define the *tangent cone* $C_{[\tilde{q}]}Q_0$ at $[\tilde{q}] \in Q_0 - Q_0^*$. The tangent cone can be defined as the limiting positions of all arcs in Q_0 linking $[\tilde{q}]$ and $[q]$, $[q] \in Q_0^*$. This definition is used in [3] for singular points of a zero-level constrained set. Take an ordinary cone for example, the tangent cone at the tip is obviously all (3-D) vectors at the tip that lie outside the region enclosed by the cone and the negative cone obtained by extending the cone beyond the tip. Higher order tangent cones are similarly defined, with the help of TQ_0^*, and so on.

The set $C_{[\tilde{q}]}Q_0$ of tangent cone vectors, as it is defined thus far, may not be a vector space. However, due to the embedding mentioned above, we can borrow the topology of the embedding space and define the tangent space at the point concerned to be the linear span of the tangent cone at that point [3]. The resulting tangent space, which we denote as $T^C_{[\tilde{q}]}Q_0$, in general has a higher dimension than the quotient space. This way we can define a tangent bundle on Q_0. Higher order bundles can be similarly defined, utilizing the embedding.

The Whitney-smooth functions $C^\infty(Q_0)$ are evidently compatible with this definition of tangent bundles. It is also evident that the maps which we called Whitney-smooth is also infinitely differentiable with respect to these bundles.

Note that since Q_0^* is dense in Q_0, a map τ is Whitney-smooth on Q_0 if it is defined by extending some smooth map defined on Q_0^*.

2.2. *The quotient space Q_0 as a generalized transverse sub-manifold*

From Section 2.1, the root cause of the non-smoothness of Q_0 is the existence of multiple orbit types in Q. This is why the assumption of single orbit type is very common in many techniques in this field. We shall demonstrate how they can be generalized to multiple orbit type situations by generalizing Helgason's analysis on invariant operators. The central idea is to construct some space M with a single orbit type, on which standard techniques apply; then pull the results back to Q.

For simplicity of presentation, we shall first assume that the dense Riemannian manifold Q_0^* is diffeomorphic to some transverse sub-manifold of Q^*, also denoted by Q_0^*, all points of which share the same isotropy group T (of maximal type of course), and the closure of which intersects any orbit once and only once. This is met by all the examples we shall encounter. The general case without the simplification is treated in Appendix A.

Given the assumption, we can identify the quotient manifold Q_0 with the closure of the sub-manifold specified above. Let the identification map be τ. It is a diffeomorphism when restricted to the dense subset Q_0^*. By the property of quotient maps and assumptions, the map τ obviously is the closure of $\tau|_{Q_0^*}$, i.e. for any sequence $[q_n] \in Q_0^*$ such that $\lim_{n\to\infty}[q_n] = [\tilde{q}] \in Q_0 - Q_0^*$,

$$\lim_{n\to\infty} \tau([q_n]) = \tau([\tilde{q}]). \tag{2.5}$$

Such a map τ is therefore Whitney-smooth on Q_0.

In fact we have, for any $[q] \in Q_0^*$, and $\tau([q]) = q \in Q^*$,

$$T\tau(T_{[q]}Q_0^*) = (T_q(G \cdot q))^{\perp}. \tag{2.6}$$

For any non-maximal point $[\tilde{q}]$, $\tau([\tilde{q}]) = \tilde{q} \in Q - Q^*$, and vector $X_{[\tilde{q}]} \in C_{[\tilde{q}]}Q_0$,

$$T\tau(X_{[\tilde{q}]}) = \lim_{t \to \infty} \frac{\tau([\tilde{q}] + X_{[\tilde{q}]}t) - \tilde{q}}{t}, \tag{2.7}$$

where $[\tilde{q}] + X_{[\tilde{q}]}t \in Q_0^*$, by definition of tangent cones. The above limit exists since τ is Whitney-smooth. Note that (2.7) is nothing more than a defining equation for the tangent cone at $\tau([\tilde{q}]) = \tilde{q}$ in the set $\tau(Q_0)$ embedded in the manifold Q. Evidently, $T\tau$ maps the tangent cone at $[\tilde{q}]$ onto the tangent cone at $\tilde{q}$, which must be contained in the space $(T_{\tilde{q}}(G \cdot \tilde{q}))^{\perp}$. Recall that the conical tangent space at $\tilde{q}$ is the linear span of the cone. Therefore we have $T\tau(T^C_{[\tilde{q}]}Q_0) \subseteq (T_{\tilde{q}}(G \cdot \tilde{q}))^{\perp}$.

Notice that the map $T(T\tau)$ can be defined similarly on the second order bundle $T^C(T^C Q_0)$, and so on.

The map $T\tau$ induces a metric, denoted $\tilde{g}$, on Q_0 by

$$\tilde{g}(X_{[q]}, Y_{[q]}) = g(T\tau(X_{[q]}), T\tau(Y_{[q]})), \quad X_{[q]}, Y_{[q]} \in T^C_{[q]}Q_0. \tag{2.8}$$

The induced metric is non-degenerate on Q_0^*, and may be degenerate on $Q_0 - Q_0^*$. The degeneracy will not affect the subsequent computation. Note that the metric so defined, when restricted to each stratum of Q_0, coincides with the already well-defined reduced metric on that stratum.

Remark. In general, we can only expect local diffeomorphisms from Q_0 into Q. But the G-invariance of the metric g still enables us to define a *global* metric on Q_0 (cf. Appendix A.1).

Now we construct a direct product space M, which can be mapped Whitney-smoothly onto Q. Such spaces will play a central role in our discussion. Let P be the homogeneous space associated with maximal orbits (i.e. P is diffeomorphic to some G/G_q, $q \in Q^*$). Let us form the space $M = Q_0 \times P$. Let M carry the (transitive) action of G,

$$g \cdot ([q], p) = ([q], L_g \cdot p), \quad g \in G,\ [q] \in Q_0,\ p \in P. \tag{2.9}$$

On M, there is only one orbit type. All the standard techniques apply on M, all we need is a way to pull the results back to Q.

The tangent space (or cone) on M by definition is the direct sum of the tangent space (or cone) on Q_0 and the tangent space on P.

Proposition 1. *There is a Whitney-smooth map π from $M = Q_0 \times P$ onto Q.*

Proof. As before, let τ be the Whitney-smooth map from Q_0 into Q. Let κ be a diffeomorphism identifying P with $G/G_{\tau([q])}$. Denote $G_{\tau([q])}$ now as H. Note that for any $[q] \in Q_0^*$, we can identify $(\tau([q]), \kappa(P))$ with $G \cdot q$ via a diffeomorphism π', given by $g_H \cdot \tau([q])$, $g_H \in \kappa(P)$ [7,13]. The overall map from $[q] \times P$ to $G \cdot q$ is then denoted as $\pi' \circ (\tau, \kappa)$.

Since H acts trivially on $\tau(Q_0^*)$, $\pi' \circ (\tau, \kappa)$ is well-defined for all points in Q_0^*. Hence the definition of κ is independent of $[q]$. Thus $\pi' \circ (\tau, \kappa)$ is globally a diffeomorphism. For $[\tilde{q}] \in (Q_0 - Q_0^*)$, by definition $\tau([\tilde{q}]) = \tilde{q}$ is the limit of some sequence $\{\tau([q_n])\}$. Further, $G_{\tilde{q}}$ contains H. Therefore, the map $\pi' \circ (\tau, \kappa)$ is well-defined even on $(Q_0 - Q_0^*) \times P$ as the closure of the map on $M^* = Q^* \times P$. Hence the map $\pi' \circ (\tau, \kappa)$ is Whitney-smooth from M onto Q. Let us denote it as π. □

We sometimes refer to this map as a projection since it is reminiscent of the projection from a plane onto a sphere. The projection, being Whitney-differentiable, pushes tangent spaces of M onto some corresponding tangent spaces on Q by $T\pi$. In fact, for any $([q], p) \in M^*$ so that $\pi(([q], p)) = q \in Q$, $T\pi = T\pi'|_{(\tau([q]),\kappa(p))} \circ (T\tau|_{[q]}, T\kappa|_p)$, one has

$$T\pi(T^C_{[q]}Q_0 \oplus T_p(P)) = (T_q(G \cdot q))^{\perp} \oplus T_q(G \cdot q), \tag{2.10}$$

where $T^C_{[q]}Q_0 = T_{[q]}Q_0$ here. The first component comes from the map $T\tau$ acting on the tangent spaces of each $Q_0 \times \{p\}$, and the second component is obvious from the definition of π' above. For any $([\tilde{q}], p) \in M - M^*$ with $\pi(([\tilde{q}], p)) = \tilde{q} \in Q$, one has

$$T\pi(T^C_{[\tilde{q}]}Q_0 \oplus T_p(P)) = T\tau(T^C_{[\tilde{q}]}Q_0) \oplus T_{\tilde{q}}(G \cdot \tilde{q}). \tag{2.11}$$

The map $T\pi$ always has non-empty kernel on points in $M - M^*$ since $\tilde{q}$ is non-maximal so $\dim T_{\tilde{q}}(G \cdot \tilde{q}) < \dim T_p(P)$. Also, the map π when restricted to M^* is a diffeomorphism between M^* and Q^*.

For any $p \in P$, the space $Q_0 \times \{p\}$, denoted as (Q_0, p), satisfies the following conditions: for any $([q], p) \in (Q_0, p)$,

(a) $G \cdot ([q], p) \cap (Q_0, p) = \{([q], p)\}$,

(b) $T^C_{([q],p)}M = T^C_{([q],p)}((Q_0, p)) \oplus T_{([q],p)}(G \cdot ([g], p))$. (2.12)

These conditions would define 'transverse' sub-manifolds [14]. The dense subset (Q_0^*, p) is indeed transverse in M^* in the conventional sense. We shall extend this notion by calling (Q_0, p) transverse (in M), although Q_0 may contain singularities. In the more general setting of the appendix, we have the same extended concept of transversality.

2.3. The induced measure on M

The projection π induces a metric on M, defined by $g \circ T\pi$, where g is the original metric on Q. Let us also denote the induced metric by g.

Remark. In general, the projection π can only be locally smooth, however, we still can induce a metric by the G-invariance of this metric (cf. Appendix A).

Since $M = Q_0 \times P$ is a direct product space, we have

$$T^C_{([q],p)}M = T^C_{[q]}Q_0 \oplus T_pP.$$

Hence

$$g([q], p) = \hat{g}([q], p) \oplus \gamma([q], p). \tag{2.13}$$

From (2.8), (2.10), (2.11), $\hat{g}$ must in fact coincide with $\tilde{g}$, which depends only on $[q]$. From (2.11), the induced metric g is degenerate on $M - M^*$. The dense subspace M^* is a Riemannian manifold diffeomorphic to Q^*.

Let $\mathrm{d}[q]$ denote the Riemannian measure from $\tilde{g}([q])$, and let $\mathrm{d}\sigma([q], p)$ denote the measure from $\gamma([q], p)$. The induced measure $\mathrm{d}m$ on M is therefore written as $\mathrm{d}[q]\,\mathrm{d}\sigma([q], p)$, which satisfies $\mathrm{d}[q]\,\mathrm{d}\sigma([q], p) = \mathrm{d}q \circ \pi$. Consider $q \in Q^*$. On the one hand, the orbit $G \cdot q \subset Q^*$ inherits a Riemannian structure from Q. The corresponding Riemannian measure is of course invariant under G. The diffeomorphism identifying $G \cdot q$ with the manifold $\{[q]\} \times P$, denoted $([q], P)$, gives the metric structure γ and the corresponding measure $\mathrm{d}\sigma([q], p)$ to $([q], P)$. On the other hand, $([q], P)$ can be identified with the space P. However, P has a unique invariant measure $\mathrm{d}p$, up to a constant. The uniqueness then implies that $\mathrm{d}\sigma([q], p) = \rho([q])\,\mathrm{d}p$, where ρ is called density function [14]. The argument can be applied to the manifold $([\tilde{q}], P)$, $[\tilde{q}] \in Q_0 - Q_0^*$. Note then that $\rho([\tilde{q}])$ must be zero for $[\tilde{q}] \in Q_0 - Q_0^*$, because of the degeneracy of the induced metric $\gamma([\tilde{q}], p)$.

2.4. *The radial part of the Laplacian on Q*

The first consequence of the discussion is that the Hilbert space $L^2(Q, \mathrm{d}q)$ is naturally unitarily equivalent to the Hilbert space $L^2(M, \mathrm{d}v)$, which in turn is naturally equivalent to $L^2(Q_0, \rho([q])\,\mathrm{d}[q]) \otimes L^2(P, \mathrm{d}p)$. To see this, let us take the set $C_c^\infty(Q^*)$ of compactly supported smooth functions on Q^*, which is dense in $L^2(Q^*, \mathrm{d}q)$. Under the pull-back π^* of the projection π, the image of $C_c^\infty(Q^*)$ is denoted as $\pi^* C_c^\infty(Q^*)$. Since π is a diffeomorphism on M^*, $\pi^* C_c^\infty(Q^*)$ is simply $C_c^\infty(M^*)$. The map preserves both the L^2-norm and the supremum ($\|\cdot\|_\infty$) norm. We can then extend π^* to $L^2(Q^*, \mathrm{d}q)$. The statement then follows due to the fact that $L^2(Q^*, \mathrm{d}q) = L^2(Q, \mathrm{d}q)$ and $L^2(M, \mathrm{d}v) = L^2(M^*, \mathrm{d}v)$.

Note that $L^2(Q, \mathrm{d}q)$ is the quantum unconstrained state space (Section 1.2). We shall now transform all the essential physical objects into $L^2(M, \mathrm{d}m)$.

Under the isomorphism π^*, the original Laplacian Δ is transformed to Δ_*

$$\Delta_* \circ \pi^* = \pi^* \circ \Delta. \tag{2.14}$$

Recall that Δ is defined on $C_c^\infty(Q)$, and such Δ is e.s-a. By definition, Δ_* is e.s-a. when defined on $\pi^* C_c^\infty(Q)$.

Remark. The space $\pi^* C_c^\infty(Q)$ and $\pi^* C^\infty(Q)$ are vector spaces isomorphic to $C_c^\infty(Q)$ and $C^\infty(Q)$, respectively. They play an important role in our calculation. The smooth structure $C^\infty(M)$ on M can often be intrinsically defined, such as the Whitney-smooth structure. In that case, $\pi^* C^\infty(Q)$ will be a subset of it.

It is not hard to see that when restricted to M^*, such a Hamiltonian Δ_* is precisely the Laplacian defined by the metric induced on M^*. We cannot simply extend such a Laplacian to the entire M since the induced metric is degenerate on $M - M^*$. If we formally define it according to Eq. (2.2), the expression of such a Laplacian is meaningless on points in $M - M^*$. However, by defining the Laplacian as in (2.14), we in effect find a space of functions, namely $\pi^* C^\infty(Q)$, on which the action of the Laplacian defined by the degenerate metric of M remains meaningful even on points in $M - M^*$. One special case of this is the ordinary Euclidean Δ and its polar coordinate form.

We have extended the concepts of 'transverse' sub-manifold to include subsets such as $(Q_0, p_0) \subset M$, where p_0 is some given point on P. Write (Q_0, p_0) as Q_0 for convenience. We can also define the radial part Δ_*^0 of Δ_* as

$$\Delta_*^0 f|_{Q_0} = (\Delta_* f)|_{Q_0}, \tag{2.15}$$

where f is any invariant (or locally invariant) function belonging to $\pi^* C^\infty(Q)$. We denote the set of such functions by $\pi^* C^\infty(Q)^G$. (Note that since the projection π from M onto Q is equivariant, we have $(\pi^* C^\infty(Q))^G = \pi^*(C^\infty(Q)^G)$.) So the newly defined operator Δ_*^0 is naturally defined for functions in $\pi^* C^\infty(Q)^G|_{Q_0}$, which coincides with the space $C^\infty(Q_0)_{(0)}$ defined by Eq. (2.4) in Section 2.1. Note well that only $\pi^* C^\infty(Q)^G|_G$ is well-defined in general, while $C^\infty(Q)^G|_{Q_0}$ is only defined when Q_0 can be smoothly embedded in Q. See Section A.1.

Proposition 2. *The radial part of the Laplacian Δ (Δ_*) is*

$$\Delta_*^0 = \rho^{-1/2} \Delta_{Q_0} \circ \rho^{1/2} - \rho^{-1/2} \Delta_{Q_0}(\rho^{1/2}), \tag{2.16}$$

where Δ_{Q_0} is the Laplacian given by the metric $\tilde{g}$ in (2.8). Further, it has e.s-a. domain $C^\infty(Q_0)_{(0)} = \pi^* C^\infty(Q)^G|_{Q_0}$.

The proof of Theorem 3.7. of Chapter II.3 in [14] essentially applies to the first part of the proposition. The only difference is that our operators Δ_* and Δ_*^0 are not defined on the standard smooth function space which is no longer defined since M and Q_0 are not smooth manifolds. Instead they are defined on $\pi^* C^\infty(Q)$ and $C^\infty(Q_0)_{(0)}$, respectively. In all other respect, the proof of [14] goes as before. The second part follows easily from the definition of Δ_* and $\pi^* C^\infty(Q)$ and the e.s-a.-ness of Δ on $C^\infty(Q)$.

2.5. *Application to Rieffel induction*

To see the relevance of the 'radial part' to our quantization procedure, we now quantize classical reduction at zero-level. The zero-level constraint is quantized by the trivial representation of G on $\mathcal{H}_\chi = \mathbb{C}$. Eqs. (1.6)–(1.10) apply.

The reduced space then is $\mathcal{H}^\chi = P_{\mathrm{Id}} L^2(Q, \mathrm{d}q)$. We can unitarily transform this space by π^*. Note that π^* and P_{Id} commute, because the projection π is equivariant, and that $P_{\mathrm{Id}} f(g) = \int_g \mathrm{d}_g\, u(g) f(g)$ (cf. Section 1.2). We thus have $\pi^* P_{\mathrm{Id}} L^2(Q, \mathrm{d}q) = P_{\mathrm{Id}} \pi^* L^2(Q, \mathrm{d}q)$. But $\pi^* L^2(Q, \mathrm{d}q) \simeq L^2(Q_0, \rho([q])\, \mathrm{d}[q]) \otimes L^2(P, \mathrm{d}p)$ as shown in Section 2.4, immediately, the reduced (physical) quantum state space can be identified with $L^2(Q_0, \rho([q])\, \mathrm{d}[q])$.

Next, consider the reduced Hamiltonian Δ^0, which is, by (1.9), equal to $P_{\mathrm{Id}} \Delta$.

Apply the unitary transformation π^* to (1.9); we have
$\pi^* \Delta^0 = \pi^*(P_{\mathrm{Id}} \Delta) = P_{\mathrm{Id}}(\pi^* \Delta) = P_{\mathrm{Id}} \Delta_*$.

Recall that Δ has e.s-a. domain $C_c^\infty(Q)$. By (1.10), the operator Δ^0 is defined on $P_{\mathrm{Id}}C_c^\infty(Q)$ and that it is e.s-a. Under the isomorphism π^*, we have that $\pi^*\Delta^0$ is defined on $\pi^*P_{\mathrm{Id}}C_c^\infty(Q)$, on which it is e.s-a. (Note that all this is consistent with applying Section 1.2 directly to $\pi^*L^2(Q, \mathrm{d}q)$ and $\pi^*\Delta$ defined on $\pi^*C_c^\infty(Q)$.)

The operator P_{Id} turns functions into G-invariant functions cf. Section 1.2. This means that $P_{\mathrm{Id}}(\pi^*C_c^\infty(Q)) = \pi^*C_c^\infty(Q)|_{Q_0} = C_c^\infty(Q_0)_{(0)}$. This is an e.s-a. domain for $\pi^*\Delta^0$, cf. (1.10). One glance at the definition (2.15) of the radial part will reveal that $\pi^*\Delta^0$ is in fact the operator Δ_*^0 worked out earlier. By Proposition 2, an e.s-a. domain of Δ_*^0 is

$$D_{\text{e.s-a.}}(\Delta_*^0) = C_c^\infty(Q_0)_{(0)}, \tag{2.17}$$

agreeing with the computation by projection P_{Id}.

3. Examples

3.1. Free particle on a plane

This simple example involves the additive group $\mathbb{R}$. Sniatycki–Weinstein [30] and Arms–Gotay–Jennings [3] both treated the problem, but their results differ. Let us discuss the difference in the context of our treatment.

Let the configuration space be $Q = \mathbb{R}^2$. The phase space is then $S = T^*\mathbb{R}^2$ with canonical coordinates (q, p). $\mathbb{R}$ has an action defined by $t \cdot p = (q + tp, p)$. The momentum map is then $J(q, p) = \frac{1}{2}|p|^2 = \frac{1}{2}(p_1^2 + p_2^2)$ (i.e. the energy). The constraint is $J = \frac{1}{2}m^2$. The classical reduction of this simple system is already tricky. Note that at $m^2 = 0$ the dimension of $J^{-1}(m^2)$ suddenly drops from 3 to 2. The reduction at $m^2 = 0$ is singular.

According to geometric reduction (a generalization of symplectic reduction cf. [3] to incorporate singular systems), the phase space of the system reduces to T^*S^1 when $m^2 > 0$, and to a point $\{0\}$ when $m^2 = 0$ [3]. This result makes physical sense; also note the sudden drop of dimensions here.

Sniatycki and Weinstein [30] treated this system in the context of algebraic reduction, which they developed in order to quantize singular systems. It is a procedure to produce a reduced classical observable algebra. In this example, it gives non-trivial observable algebra for both $m^2 > 0$ and $m^2 = 0$, contradicting the result of [3] at $m^2 = 0$.

According to the method of Sniatycki and Weinstein the system at $J = 0$ is quantized by a system whose state space is made up by the solutions of the equation $-(\partial^2{}_{q_1} + \partial^2{}_{q_2})\psi^q = 0$, or $(p_1^2 + p_2^2)\psi^p = 0$ in p-space. The solutions involve Dirac delta 'functions' which span the solution space. The use of Dirac delta functions leads to complications.

We quantize the system by Rieffel induction. First, the unconstrained system: the unconstrained quantum phase space is $L^2(\mathbb{R}^2)$; the unitary representation is $U(t) = \mathrm{e}^{(\mathrm{i}|p|^2t/2)}$; the classical co-adjoint orbits are single points which are quantized by $\mathcal{H}_{m^2} = \mathbb{C}$ with $\pi_{m^2}(t) = \mathrm{e}^{-\mathrm{i}m^2t/2}$.

For our induction procedure, cf. Section 1.2, choose the dense subspace $L = C_c(\mathbb{R}^2)$. The rigged inner product on $L \otimes \mathcal{H}_{m^2} = L \otimes \mathbb{C} \simeq L$, by (1.5), is

$$(\psi_1, \psi_2)_0 = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^2} \mathrm{d}p e^{\mathrm{i}|p|^2 t} \psi_1(p)\overline{\psi_2}(p) e^{-\mathrm{i}m^2 t},$$

where $\psi_i \in L$. The rigged inner product $(\, , \,)_0$ can be evaluated on $L = C_c(\mathbb{R}^2)$. The result (valid for all m^2) is

$$(\psi, \psi)_0 = (\pi/2) \int_0^{2\pi} \mathrm{d}\theta |\psi(|p|^2 = m^2)|^2,$$

where $\psi(|p|^2 = m^2)$ evidently depends only on the direction θ. Hence the null-space $\mathcal{N}_0 = \{\psi \in L | \psi(|p|^2 = m^2) = 0\}$.

This implies that the reduced phase space $\mathcal{H}^{m^2}$ (the completion of $L/\mathcal{N}_0$) is $L^2(S^1, \mathrm{d}\theta)$ when $m^2 > 0$. However, when $m^2 = 0$ the change is dramatic: here $\mathcal{H}^{\chi} = \mathbb{C}$, which is different from the result of [30]. This mirrors the drop of dimensions in the classical geometric reduction.

Note firstly that for any m^2, the reduced spaces turn out to be the direct quantization of the result of geometric reduction. *Rieffel induction quantizes the geometric, rather than the algebraic reduction procedure.* Secondly there is no problem about normalization of wave functions. We usually encounter this problem whenever we deal with quantum systems on non-compact configuration spaces. The usual way to deal with it is to introduce the Dirac delta function. It is what Sniatycki and Weinstein did. Here we showed a perhaps cleaner alternative.

The calculation is immediately generalizable to any dimension.

3.2. Angular momentum

3.2.1. Direct computation with Rieffel induction

We follow [3] for the classical reduction. Let $G = SO(n)$ act on $Q = \mathbb{R}^n$ by rotation: $a \in SO(n)$ acts on $X = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ by $X \mapsto aX$. (When $n = 1$, the group is simply reflection about the origin, for which the discussion still applies.) The action is lifted to $S = T^*\mathbb{R}^n$: if $(p_1, \ldots, p_2)$ are the dual coordinates, $a \cdot (X, Y) = (aX, aP)$. With respect to the standard basis of the dual Lie algebra of $SO(n)$, $\mathfrak{g}^* \cong \mathbb{R}^n$, the momentum map J has components $J_{jk} = x_j p_k - x_k p_j$, $1 \leq j < k \leq n$.

The reduction at $J \neq 0$ is regular. In that case, the reduced phase space is diffeomorphic to $T^*\mathbb{R}^+$. However, when $J = 0$ we have a typical singular reduction situation. The reduced phase space is given by the cone $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the discrete subgroup of $SO(2)$ formed by the identity and rotation by π. The reduced configuration space $Q_0 = Q/G$, which is $[0, \infty)$, is denoted as $\overline{\mathbb{R}^+}$. It has a singularity which is the origin $\{0\}$ (but due to the low dimension it appears as a boundary).

Take $n = 2$ for example [19]. The equality $J = 0$ implies $x_1 p_2 = x_2 p_1$. Thus, the constraint surface is $M = J^{-1}(0) = \{(X, P) \in T^*\mathbb{R}^2 \mid X \| P\}$. The subset defined by $x_2 = p_2 = 0$ intersects any orbit at exactly two points. This and the definition of M imply that these two points must be related by a rotation by π. Hence the reduced phase space is $S_0 = J^{-1}(0)/G = \mathbb{R}^2/\mathbb{Z}_2$, which is a cone.

This classical system certainly falls into the category treated in Section 2. The set of maximal orbits, denoted as Q_0^* in Section 2, is $\mathbb{R}^+$, which is self-evidently diffeomorphic to a (transverse) sub-manifold of $Q = \mathbb{R}^n$. This diffeomorphism carries over to the phase spaces: $T^*\mathbb{R}^+$ is diffeomorphic to $S_0 - \{0\}$.

The standard way to quantize this system at $J = 0$ is to apply some quantization procedure for the maximum stratum [9]. This results in a reduced quantum phase space of $L^2(\mathbb{R}^+)$ and a reduced Hamiltonian defined on domain $C^\infty(\mathbb{R}^+)$. For $n < 4$, the Hamiltonian defined this way is not e.s-a., implying the necessity to study the family of s-a. extensions [9].

We shall now quantize this system with Rieffel induction. All notations follow the conventions set up so far. For concreteness, we put $n = 3$. The case of general n is virtually the same.

We start from the quantized unconstrained system, which is

$$\mathcal{H} = L^2(\mathbb{R}^3), \qquad U(a)\psi = \psi \circ a^{-1},$$

where $\psi \in \mathcal{H}$, $a \in G = SO(3)$.

The classical constraint may be any particular co-adjoint orbit $\mathcal{O}_{m^2} \in \mathfrak{g}^*$ labelled by $m^2 \in \overline{\mathbb{R}^+}$, which has to be $l(l+1)$ ($l = 0, 1, 2, \ldots$); otherwise quantization is not defined, cf. [1]. The orbit $\mathcal{O}_{l(l+1)}$ is quantized by the irreducible unitary representation $(U_l, \mathcal{H}_l)$ of $SO(3)$. We choose the standard carrier space $\mathbb{C}^{2l+1} = \mathcal{H}_l$, isomorphic to the space spanned by the lth family of spherical harmonics.

By Section 1.2, the reduced quantum state space $\mathcal{H}^l$ is $P^l_{\mathrm{Id}}(\mathcal{H} \otimes \mathbb{C}^{2l+1})$, where $P^l_{\mathrm{Id}} = \int_G \mathrm{d}g\, U \otimes \pi_l$, an orthogonal projection from $\mathcal{H} \otimes \mathbb{C}^{2l+1}$ onto the trivially transforming subspace.

Since $\mathcal{H} = L^2(\mathbb{R}^3) \simeq L^2(\mathbb{R}^+, r^2\,\mathrm{d}r) \otimes (\bigoplus_{l'} \mathbb{C}^{2l'+1})$, U can be decomposed as $\mathbb{I} \otimes \bigoplus_{l'} \pi_{l'}$. We then have $\mathcal{H} \otimes \mathbb{C}^{2l+1} \simeq \mathcal{L}^2(\mathbb{R}^+) \otimes (\bigoplus_{l'} \mathbb{C}^{2l'+1}) \otimes \mathbb{C}^{2l+1}$. However, $\mathbb{C}^{2l'+1} \otimes \mathbb{C}^{2l+1} = \otimes_{L=|l'-l|}^{l'+l} \mathbb{C}^{2L+1}$ by well-known properties of $SO(3)$. The space contains a trivially transforming subspace $\mathbb{C}$ only when $l' = l$.

Hence the reduced phase space is

$$\mathcal{H}^l = P^l_{\mathrm{Id}}(\mathcal{H} \otimes \mathbb{C}^{2l+1}) = L^2(\mathbb{R}^+) \otimes \mathbb{C}_{(l)} \simeq L^2(\mathbb{R}^+, r^2\,\mathrm{d}r), \tag{3.1}$$

where $\mathbb{C}_{(l)} \simeq \mathbb{C}$ is the trivially transforming subspace.

Next, we work out the reduced Hamiltonian. We shall compute only the free Hamiltonians, since the application to other cases is similar. The unconstrained free Hamiltonian $-\Delta$ has

$$\text{e.s-a. domain } D_{\text{e.s-a.}} = C_c^\infty(\mathbb{R}^3), \qquad \text{s-a. domain } D_{\text{s-a.}} = \{\psi \in \mathcal{H} | \Delta\psi \in \mathcal{H}\}. \tag{3.2}$$

where $\Delta\psi$ is in the distributional sense.

By (1.10), the reduced Hamiltonian is given by restriction to the subspace (3.1), which results in

$$-\Delta_r^l = -\left(D_r^2 + \frac{n-1}{r}D_r - \frac{l(l+1)}{r^2}\right) \tag{3.3}$$

with domain $D^l_{\text{e.s-a.}} = P^l_{\text{Id}}(D_{\text{e.s-a.}} \otimes \mathbb{C}^{2l+1})$ or $D^l_{\text{s-a.}} = P^l_{\text{Id}}(D_{\text{s-a.}} \otimes \mathbb{C}^{2l+1})$. It is a short exercise to verify this directly, and to check that $-\Delta_r^l$ on $D^l_{\text{s-a.}}$ is indeed the closure of $-\Delta_r^l$ on $D^l_{\text{e.s-a.}}$

3.2.2. Applying techniques in Section 2 and compare

Let us apply Section 2 and see how it works out. First we need to form the space $M = Q_0 \times P$, where P is the homogeneous space of the maximal orbits. In this case, $P = S^2$ and $Q_0 = \overline{\mathbb{R}^+}$, so $M = \overline{\mathbb{R}^+} \times S^2$. The smooth projection π from M onto $Q = \mathbb{R}^3$) is simply $\pi : (r, \Omega) \in \overline{\mathbb{R}^+} \times S^2 \mapsto x = \pi(r, \Omega) \in \mathbb{R}^3$ (i.e. the polar to Cartesian coordinate transform, where the spherical angle Ω parametrizes S^2). As expected, π restricted to $M^* = \mathbb{R}^+ \times S^2$ provides a diffeomorphism from M^* to $Q^* = \mathbb{R}^3 - \{0\}$. The induced metric on M is the usual polar form of the Euclidean metric. It is degenerate on $(0, \Omega) \in M - M^*$. The decomposition (2.13) of the metric holds, where the induced metric on Q_0 is the Euclidean metric on $\overline{\mathbb{R}^+}$. The factorisation of the measure $\mathrm{d}m$ into $\rho([q])\,\mathrm{d}[q]\,\mathrm{d}p$ is the usual $r^2\,\mathrm{d}r\,\mathrm{d}\Omega$.

By Section 2.5, the reduced space is $L^2(\overline{\mathbb{R}^+}, r^2\,\mathrm{d}r)$, which agrees with the earlier result. The unitarily transformed Hamiltonian $-\Delta_*$ under π^*, as defined in (2.14), is the usual Laplacian in polar form. Its e.s-a. domain is, by definition, $\pi^* C_c^\infty(\mathbb{R}^3)$. The reduced Hamiltonian at constraint-level zero is computed from (2.16), which results in $D_r^2 + ((n-1)/r)D_r$, agreeing with the independently obtained $-\Delta_r^0$ in (3.3). The e.s-a. domain is $C_c^\infty(\overline{\mathbb{R}^+})_{(0)}$, by (2.17).

But by (1.2), the e.s-a. domain at any constraint-level can also be written as

$$D^l_{\text{e.s-a.}} = P^l_{\text{Id}}(C_c^\infty(\mathbb{R}^3) \otimes \mathbb{C}^{2l+1}). \tag{3.4}$$

Let us see how the two domains reconcile with each other. Any $\pi^* C_c^\infty(\mathbb{R}^3)$ function can be approximated in $\|\cdot\|_\infty$ norm by functions in $C_c^\infty(\overline{\mathbb{R}^+}) \otimes \bigoplus_l^F \mathbb{C}^{2l+1}$. The superscript F indicates that only finite combinations are taken (this follows from the Peter–Weyl approximation theorem for compact groups). These approximating expansions form the set

$$\pi^* C_c^\infty(\mathbb{R}^3) \cap \left(C_c^\infty(\overline{\mathbb{R}^+}) \otimes \left(\bigoplus_l^F \mathbb{C}^{2l+1}\right)\right),$$

which we denote as

$$\bigoplus_l^F C_c^\infty(\overline{\mathbb{R}^+})_{(l)} \otimes \mathbb{C}^{2l+1}. \tag{3.5}$$

The intersection with $\pi^* C_c^\infty(\mathbb{R}^3)$ imposes conditions (which depend on l) on $C_c^\infty(\overline{\mathbb{R}^+})$, hence the subscript '$(l)$'. For instance, the fact that all $\pi^* C_c^\infty(\mathbb{R}^3)$ functions must be constants at $r = 0$ implies that functions in $C_c^\infty(\overline{\mathbb{R}^+})_{(l\neq 0)}$ must vanish at $r = 0$. At $l = 0$, by definition, the set $C_c^\infty(\overline{\mathbb{R}^+})_{(0)}$ coincides with the set $C_c^\infty(Q_0)_{(0)}$ defined in (2.4).

3.2.3. An explicit e.s-a. domain for the reduced Hamiltonians

To summarize, we have

$$\bigoplus_l^F C_c^\infty(\overline{\mathbb{R}^+})_{(l)} \otimes \mathbb{C}^{2l+1} \overset{\text{dense}}{\subset} \pi^* C_c^\infty(\mathbb{R}^3) \overset{\text{dense}}{\subset} L^2(\overline{\mathbb{R}^+} \times S^2) \cong L^2(\mathbb{R}^3).$$

The first 'dense' is in the $\|\cdot\|_\infty$ norm, while the second one is in the $\|\cdot\|_{L^2}$ norm.

In fact, $-\Delta_*$ defined on $\bigoplus_l^F C_c^\infty(\overline{\mathbb{R}^+})_{(l)} \otimes \mathbb{C}^{2l+1}$ is also e.s-a. and has the same s-a. extension as $-\Delta_*$ defined on $\pi^* C_c^\infty(\mathbb{R}^3)$. We shall call this domain $D^{\min}_{\text{e.s-a.}}$. To prove the statement we only need to show that for any $\psi \in \pi^* C_c^\infty(\mathbb{R}^3)$, there is a sequence ψ_i in $\bigoplus_l^F C_c^\infty(\overline{\mathbb{R}^+})_{(l)} \otimes \mathbb{C}^{2l+1}$ such that

$$\psi_i \to \psi \quad \text{and} \quad -\Delta_*\psi_i \to -\Delta_*\psi \text{ in the } L^2\text{-norm}.$$

The second convergence follows immediately from the fact that the first convergence is in the $\|\cdot\|_\infty$ norm, and that $-\Delta_*\psi$ belongs to $\pi^* C_c^\infty(\mathbb{R}^3)$ as well (hence possessing a sequence of approximating expansions which is easily shown to be $\{-\Delta_*\psi_i\}$).

The explicit form of $D^{\min}_{\text{e.s-a.}}$ is very complicated, and differs for different dimensions. Fortunately, we know that *an operator that contains an e.s-a. operator while it itself is contained in the (s-a.) extension of that e.s-a. operator must also be e.s-a. and have the same s-a. extension.* Hence we have a vast number of choices of explicit e.s-a. domains that lead to the same original (s-a.) Hamiltonian. For instance, we can choose

$$\begin{aligned} D_{\text{e.s-a.}} = {}& (C_c^\infty(\overline{\mathbb{R}^+})|_{\phi'(0)=0} \otimes \mathbb{C}) \\ & \oplus \bigoplus_{l>0}^F \left\{\phi \in C_c^\infty(\overline{\mathbb{R}^+})\Big|^{a.\ \phi(0)=0}_{b.\ -\Delta_r^l\phi\in L^2(\mathbb{R}^+,r^{n-1}\,\mathrm{d}r)}\right\} \otimes \mathbb{C}^{2l+1}. \end{aligned} \tag{3.6}$$

The above choice is valid for all dimensions (for $n = 1$ there is only the $l = 0$ case). However, condition (b) is only necessary for $n = 2$ since (a) implies (b) for all $n > 2$.

Now plug $D_{\text{e.s-a.}}$ into (3.4), at $l = 0$ the reduced Hamiltonian is

$$-\Delta_r^0 = -\left(D_r^2 + \frac{n-1}{r} D_r\right),$$

e.s-a. on

$$C_c^\infty(\overline{\mathbb{R}^+})|_{\phi'(0)=0}$$

and at $l > 0$, it is

$$-\Delta_r^l = -\left(D_r^2 + \frac{n-1}{r} D_r - \frac{l(l+1)}{r^2}\right),$$

e.s-a. on

$$\left\{\phi \in C_c^\infty(\overline{\mathbb{R}^+})\Big|^{a.\phi(0)=0}_{b.-\Delta_r^l\phi\in L^2(\mathbb{R}^+,r^{n-1}\,\mathrm{d}r)}\right\}.$$

When $n \geq 4$, the standard choice of e.s-a. domain for all l is $C_c^\infty(\mathbb{R}^+)$. This obviously gives the same s-a. extension as ours, since it is contained in our choice of (e.s-a.) domain. When $n < 4$, the standard choice remains valid only for $l > 0$. However, for $n < 4$ at $l = 0$, this choice produces an infinite family of reduced Hamiltonians extended from that defined on $C_c^\infty(\mathbb{R}^+)$, none of which can be excluded [9]. In contrast, Rieffel induction produces a unique choice which extends to the well-known physically correct Hamiltonian, cf. [25].

3.3. Discrete group actions

The above results can be used immediately to treat systems on $\mathbb{R}^n$ with a subgroup of $SO(n)$ as symmetry group. Let us consider the case of discrete subgroup of $SO(2)$, isomorphic to $\mathbb{Z}_m$, where m is the number of distinct elements of the discrete subgroup. The classical reduced space $\mathbb{R}^2/\mathbb{Z}_m$ is a cone. Systems on this cone appear in general relativity, cf. [8], amongst others.

Classically, a discrete symmetry group can only lead to trivial momentum maps. Hence all reduction is constraint free [20]. The reduced classical phase space is therefore $T^*(\mathbb{R}^2)/\mathbb{Z}_m = T^*(\mathbb{R}^2/\mathbb{Z}_m)$ [20]. The space $\mathbb{R}^2/\mathbb{Z}_m$ is a cone with the tip at the origin, and an opening angle depending (proportionally) on m.

The effect of the classical singularity on the quantization of systems on such a cone is studied in [8,17]. The real root of the problem is that the singularity causes the formally quantized Hamiltonian to be non-essentially self-adjoint. In the standard approach, the calculation of the family of s-a. extensions should then be carried out, which is done in detail in [17], but which one of the extension is right is not clear in this approach.

Rieffel induction applies to this problem exactly as to $SO(2)$. The only difference is that the trivially transforming subspace (which will be our reduced quantum state space) is larger here.[3] The projection P_{Id} picks out not only the $L^2(\mathbb{R}^+, r^{n-1}\,\mathrm{d}r)$ factor but also all the 'angular factors' (space spanned by the spherical harmonics on S^{n-1}) that are invariant under the unitary action of $\mathbb{Z}_m$.

Consider $\mathbb{Z}_3$ for example, the group generated by rotation by $\frac{2}{3}\pi$ on $\mathbb{R}^2$. The classical reduced configuration space is a cone with opening angle $\frac{2}{3}\pi$. Applying Rieffel induction we obtain the following:

The reduced state space $\mathcal{H}^0$ is $\mathcal{H}^0 = P_{\mathrm{Id}}L^2(\mathbb{R}^2) \cong L^2(\mathbb{R}^+, r\,\mathrm{d}r) \otimes \bigoplus_l \mathbb{C}_{(3l)}$, where $\mathbb{C}_{(3l)}$ is spanned by $\psi_l(\theta) = \mathrm{e}^{\mathrm{i}3l\theta}, l \in \mathbb{Z}$. In other words, it is the L^2-space over the cone with the $\mathbb{R}^2$-measure. The reduced Hamiltonian is the $\mathbb{R}^2$-Laplacian restricted to $\mathcal{H}^0$; formally, it looks the same as the unreduced Hamiltonian, but with e.s-a. domain

$$D_{\mathrm{e.s\text{-}a.}} = (C_c^\infty(\overline{\mathbb{R}^+})|_{\phi'(0)=0} \otimes \mathbb{C}) \oplus \bigoplus_{3l}^{F} \left\{\phi \in C_c^\infty(\overline{\mathbb{R}^+})|_{b.-\Delta_r^{3l}\phi \in L^2(\mathbb{R}^+, r\,\mathrm{d}r)}^{a.\phi(0)=0}\right\} \otimes \mathbb{C}^{3l}.$$

[3] Another difference is that the quantum constraint can be any irreducible unitary representation of the discrete group. The parameters labelling the representations give rise to vacuum angles [31]. Only the trivial representation (vacuum angle = 0) is relevant to comparisons with other methods made here.

We see that the Hamiltonian is well defined. It is easy to show that the unique s-a. extension coincides with the *Friedrichs extension*, cf. [25], which is a special member of the family of extensions calculated in [17]. Our result agrees with the choice of [8], where the argument is made on physical grounds.

4. Dynamics on $G/Ad\text{-}G$

4.1. Preliminaries

Let G be a compact semi-simple, simply connected Lie group, with maximal torus T and Weyl group W. Then $G/Ad\text{-}G = T/W$, cf. [13,14]. An example $G = SU(n)$ will illustrate the ideas involved. On $SU(n)$, the adjoint action is simply matrix conjugation. Every unitary matrix can be diagonalized by conjugation. This means that the adjoint orbit of $A \in SU(n)$ must intersect the closed subgroup $S\Delta(n)$ of special diagonal matrices at least once. The closed subgroup $S\Delta(n)$ is a maximal torus T of $SU(n)$. We can work out the geometry of the quotient space $SU(n)/Ad\text{-}SU(n)$ by looking at the restriction of the Ad-action on T, which, by definition of W, is exactly the action of the corresponding Weyl group W on T. Notice that since eigenvalues are unchanged under conjugation, the conjugate action on T must have the effect of permuting the diagonal entries of matrices in T. In fact, for $SU(n)$ (and $U(n)$), $W = S(n)$, the permutation group of n elements, cf. [13,14].

Let the classical configuration space Q be G, carrying the adjoint action of G on itself. Let T^*Q carry the lifted action. This action is strongly Hamiltonian, with momentum map J [1]. Without going into details, we just remark that this system when reduced at $J = 0$, classically produces $J^{-1}(0)/G = T^*(G/Ad\text{-}G)$, which is highly singular. The manifold of G has natural G-invariant metric g and measure $\mathrm{d}g$. The unreduced Hamiltonian is defined by the invariant metric as in (2.1). We therfore have a well-posed mechanical problem, to which the techniques in Section 2 apply.

A partial reduction of Yang–Mills theory on a circle is equivalent to the above finite-dimensional unconstrained system [6,21]. A further reduction then produces the physical Yang–Mills theory, which is the above reduced system on $G/Ad\text{-}G$. Rieffel induction can quantize the 2-step reduction elegantly.[4] The quantization of the first step is done in [31]. Here, we shall 'jump' directly to quantize the second step, i.e. quantize the above finite-dimensional reduction, in order to concentrate on the singularity problem in the reduced space $G/Ad\text{-}G$.

4.2. Rieffel induction for $G/Ad\text{-}G$

We first quantize the unconstrained system: the unconstrained Hilbert space is $L^2(G, \mathrm{d}g)$, carrying the usual unitary representation $U : g \mapsto U(g)f = f \circ Ad_{g^{-1}}$; the Hamiltonian is the Laplacian Δ on the manifold G with e.s-a. domain $C^\infty(G)$.

[4] Of course, we can also equivalently quantize the complete reduction in one go [31].

All the machinery developed in Section 2 directly applies here. By Section 2.5, we obtain the physical quantum Yang–Mills theory on a circle: the physical Hilbert space is

$$L^2(G/Ad\text{-}G, \rho([g])\,\mathrm{d}[g]),$$

where $[g] \in G/Ad\text{-}G$, $\mathrm{d}[g]$ is the Lebesgue measure on $G/Ad\text{-}G$, and $(\rho([g]))^{1/2}$ is given by Eq. (63) of Chapter II.3 of [14] in terms of the root system of the group; the physical Hamiltonian is given by (2.16) with e.s-a. domain $C^\infty(G/Ad\text{-}G)_{(0)}$ defined in (2.4); the eigenvectors of the Hamiltonian lie in $C^\infty(G/Ad\text{-}G)_{(0)}$, and are the characters of the group.

The e.s-a. domain for the reduced Hamiltonian can be given more explicitly. Notice that the space $T/W = G/Ad\text{-}G$ is the so-called fundamental domain of the group, which can be identified with some compact convex subset of $\mathbb{R}^{\mathrm{rank}\,G}$. In fact, it is a rank G dimensional polyhedron. It is a stratified space, in accordance with the theory set up in Section 2. The maximal stratum is the interior. The lower-dimensional strata include the $(\mathrm{rank}\,G - 1)$-dimension hyperplanes (walls); they are the walls of Weyl chambers; the intersections of the completion of walls form *edges* of various dimensions $< (\mathrm{rank}\,G - 1)$; the lowest-dimensional 'edges' are the vertices of the polyhedron. We may call any of these strata an *edge* since walls and vertices are extreme cases of an *edge*.

Note that the walls, edges and vertices of the polyhedron are singularities as we defined in Section 2.1. For instance, vertices correspond to the smallest orbit type, the fixed points of the action of the Weyl group W on T, or equivalently, of the adjoint action of G on G; edges of increasing dimensions and walls correspond to progressively larger orbit types. The interior corresponds to the maximal type. We shall use $\partial(G/Ad\text{-}G)$ to denote these singularities. Next, in order to express our theorem succinctly, we extend the concept of *normal derivative* as follows.

Definition. At any point on an edge of dimension rank $G - 1 - d$, where $0 \le d \le (\mathrm{rank}\,G - 1)$, we form the local cylindrical coordinates $(Z, (r, \Omega))$, where Z denotes the cartesian coordinates along the edge, and (r, Ω) the polar coordinates on the $d + 1$-dimensional hyperplane normal to the edge. The normal derivative at that point is the first order radial derivative in the above local coordinates.

When $d = 0$, this coincides with the usual normal derivative; when $d = (\mathrm{rank}\,G - 1)$, the local cylindrical coordinates reduce to local radial coordinates, the normal derivative is simply the radial derivative.

We have Whitney-smooth structure $C^\infty(G/Ad\text{-}G)$ defined by restriction of smooth functions on $\mathbb{R}^{\mathrm{rank}\,G}$, cf. Section 2.1. The second smooth structure $C^\infty(G/Ad\text{-}G)_{(0)}$, by definition, contains invariant functions. Because the Weyl group acts by reflection on the walls, the invariance implies that their normal derivative, as defined above, must vanish on the walls.

Theorem 3. *For Yang–Mills theory on a circle with semi-simple simply connected compact Lie structure group G, the physical Hamiltonian Δ^0, given by* (2.16) *has e.s-a. domain* $C^\infty(G/Ad\text{-}G)|_{\mathrm{Neumann}} = \{\phi \in C^\infty(G/Ad\text{-}G) | \phi'|_{\partial(G/Ad\text{-}G)} = 0\}$, *where* $\phi'|_{\partial(G/Ad\text{-}G)}$ *is* *the generalized normal derivative, and $C^\infty(G/Ad\text{-}G)$ is the Whitney-smooth structure on $G/Ad\text{-}G$.*

Proof. If a symmetric operator contains an e.s-a. operator, then it must be e.s-a. with the same s.a extension. Denote Δ^0 defined on $C^\infty(G/Ad\text{-}G)_{(0)}$ by $T_{\text{e.s-a.}}$, and Δ^0 defined on

$$C^\infty(G/Ad\text{-}G)|_{\text{Neumann}} \supset C^\infty(G/Ad\text{-}G)_{(0)}$$

by T_s. Hence $T_s \supset T_{\text{e.s-a.}}$. It is well known that the Laplacian Δ^0 involves up to second order derivatives, and its coefficients are at most singular to the order of $\lim_{x\mapsto 0} 1/x^2$, cf. [14], while the factor $\rho([g])$ in the measure is of the order $x^{\text{rank}\, G-1}$. By simple order counting, we can show that T_s is well-defined in $L^2(G/Ad\text{-}G, \rho([g])\,\mathrm{d}[g])$, and is indeed symmetric. □

Let us now look at two explicit examples.

4.3. Rieffel induction for $SU(2)/Ad\text{-}SU(2)$

For $SU(2)$, we have $T = S(U(1)\otimes U(1)) \cong U(1)$, i.e. a circle. The action of the only non-trivial member of the Weyl group $W = S(2)$ is a reflection with respect to a chosen diameter of the circle. There are two orbit types: two fixed points and the rest of the points (maximal type) whose W-orbits consist of two points. For a point of maximal type, its $SU(2)$-orbit is diffeomorphic to $SU(2)/T$ since its stability group is T.

The reduced space $SU(2)/Ad\text{-}SU(2)$, or Q_0 in our notation, is therefore identified with an interval $\bar{I} = [0, \pi]$ with two *distinct* boundary points denoted also by e and $-e$, since they correspond to (the unit matrix) e and $-e$ of $SU(2)$. The set of maximal orbits Q_0^*, or the maximal stratum, is the interior of the interval.

In the construction in Section 2, we formed the space $M = Q_0 \times P$, which is now $\bar{I} \times SU(2)/T$. We can project M onto $SU(2)$ by $\pi : M \ni (t, a_T) \mapsto a \cdot t$, where a is any member of the orbit a_T. This identifies $M^*(= I \times SU(2)/T)$ with $Q^* = SU(2) - \{e, -e\}$. The above is obviously true from the fact that $SU(2)$ is S^3 and $SU(2)/T$ is S^2 (rather like identifying a ball without the two poles with a finite-height cylinder without the brims.)

The metric of $SU(2)$ induces a metric by the map π. To calculate the corresponding factorized measure, we need to work out the density function ρ. The simple roots of $SU(2)$ give $\rho(t) \propto \sin^2(t)$, if we choose $t \in \bar{I} = [0, \pi]$, cf. [14].

Hence from (2.16)

$$-\tfrac{1}{2}(\Delta^0) = -(D_t^2 + 2\cot(t)D_t). \tag{4.1}$$

Remark. This Hamiltonian is independently obtained formally in the context of Calogero-type integrable system, where it is called the physical Hamiltonian, and is expressed as $-\frac{1}{2}(\Delta) + V$ acting in the unreduced state space $L^2(G)$, i.e. the reduced system is seen as a result of the unreduced system with a potential [11]. The potential is called the generalized

Sutherland potential. It can be verified that the potential in fact exactly cancels out the 'angular' part of the free Hamiltonian on $SU(2)$ and leaves us with (4.1). However, the issue of its self-adjointness has not been properly discussed.

An e.s-a. domain of the reduced Hamiltonian Δ^0 is given by the Neumann boundary condition (cf. Theorem 3), i.e. $C^\infty(\bar{I})|_{\phi'(\partial \bar{I})=0}$.

Remark. The eigenfunctions of the reduced Hamiltonian are the characters of the group. In the $SU(2)$ case they are $\sin((2l+1)t)/\sin(t)$, where l takes integer or semi-integer values. Note that the characters belong to $C^\infty(\bar{I})_{\text{Neumann}}$, and they *do not satisfy the Dirichlet boundary condition*, which is the boundary condition widely assumed in the study of quantum systems on an alcove (i.e. on $G/Ad\text{-}G$). The difference in the boundary condition is known to make crucial difference in the ground state of the system [26].

4.4. *Rieffel induction for $SU(3)/Ad\text{-}SU(3)$*

For $SU(3)$, the maximal torus can be chosen as $T = S(U(1)\otimes U(1)\otimes U(1))$. The Weyl group acts by permutation of the entries of the diagonal matrices. There are three orbit types:

(1) three fixed points;
(2) points with $SU(3)$-orbit diffeomorphic to $SU(3)/S(U(2)\otimes U(1))$;
(3) points (maximal type) with $SU(3)$-orbit diffeomorphic to $SU(3)/T$.

The quotient space Q_0, i.e. $SU(3)/Ad\text{-}SU(3) = T/W$, is diffeomorphic to an equilateral triangle, which we denote as $\{\bar{\Delta}\}$: The three *distinct* vertices are the three stationary points; the three sides are points of type (2); the interior, the maximum stratum, Q_0^*, denoted by $\{\Delta\}$, are points of maximal orbit, type (3).

Thus, the physical Hilbert space is $L^2(\{\bar{\Delta}\}, \rho(t)\,\mathrm{d}t)$; the physical Hamiltonian is $-\Delta^0$, e.s-a. on $C^\infty(\{\bar{\Delta}\})_{(0)}$.

To work out the explicit expression for $-\Delta^0$, we only need $\rho(t)$, $t\in\{\bar{\Delta}\}$. It is convenient to use polar coordinates (r,θ) on Q_0. Let us choose a polar coordinate system centred at a vertex, so that $t\mapsto(r,\theta)$ with $0\le r\le\pi$ and $0\le\theta\le\frac{1}{3}\pi$. Then the Lebesgue measure on T is the usual $r\,\mathrm{d}r\,\mathrm{d}\theta$. In polar coordinates, the function $\rho(t)$ must have the form

$$\sin^2(a_0 r\sin(\theta))\sin^2(a_0 r\sin(\theta+\tfrac{1}{3}\pi))\sin^2(a_0 r\sin(\theta+\tfrac{2}{3}\pi)),$$

where a_0 is $1/\sin(\frac{1}{3}\pi)$. This can either be calculated from the simple roots of $SU(3)$ or by symmetry properties of $\rho(t)$. We can write down the reduced Hamiltonian using (2.16), up to an overall constant factor

$$-\Delta^0 = -\left(D_r^2 + \frac{1}{r}D_r + 2a_0\sum_{m=0}^{2}\sin(\theta+m\pi/3)\cot(a_0 r\sin(\theta+m\pi/3))D_r \right.$$
$$\left. + \frac{-1}{r^2}(D_\theta^2 + 2a_0 r\sum_{m=0}^{2}\cos(\theta+m\pi/3)\cot(a_0 r\sin(\theta+m\pi/3))D_\theta)\right).$$

An e.s-a. domain for the reduced Hamiltonian Δ^0 is again given by the Neumann boundary condition: $C^\infty(\overline{\Delta})|_{\phi'(\partial\overline{\Delta})=0}$.

Appendix A

A.1. Full treatment of Section 2 without the simplifying assumption

In Section 2, we made the simplifying assumption that the maximal stratum $Q_0^* = Q^*/G$ of the reduced space Q_0 is diffeomorphic to a transverse sub-manifold of the maximal stratum Q^* of Q. Let us now drop this assumption. As before, we shall indicate non-maximal points in Q_0 or in Q with a tilde. We shall always let G_q denote the isotropy group of a point q and, use '[ ]' to denote an element of the quotient space as well as an orbit.

To proceed, we need to introduce a concept that is central to our discussion. A *slice* at $q \in Q$ is a sub-manifold S_q containing q satisfying the following criteria [7,12]:

(1) S_q is closed in $G \cdot S_q$;
(2) $G \cdot S_q$ is an open neighbourhood of the orbit $G \cdot q$;
(3) $G_q \cdot S_q = S_q$;
(4) $g \cdot S_q \cap S_q \neq \emptyset$ implies that $g \in G_q$.

We will list some important properties of a slice [7,12]:

(a) Some neighbourhood of $q \in Q$ is diffeomorphic to $S_q \times B$, where B is a local cross section over G/G_q (i.e. the natural projection from G onto G/G_q restricted to B is a diffeomorphism) containing the identity e.
(b) $q' \in S_q$ implies $G_{q'} \subseteq G_q$. Under our assumption, G_q is always compact, cf. Section 2. Thus, together with Proposition 1.9 of [7], we have: If q is maximal then G_q acts trivially on S_q, and S_q intersects any orbit at most once.
(c) The slice S_q is a G_q-space on which G_q acts. A slice $S_{q'}$ at $q' \in S_q$, defined with respect to the slice S_q as a G_q-space, is also a slice at q' defined with respect to Q, a G-space, and $S_q/G_q \simeq (G \cdot S_q)/G \subset Q_0$. Further, in our context, the number of orbit types is locally finite everywhere in Q [9].

In the present context, S_q can in fact be given by [4],

$$S_q := \{\exp(X_q) | X_q \in (T_q(G \cdot q))^\perp, |X_q| < \epsilon\} \tag{A.1}$$

for suitable $\epsilon > 0$. Hence $T_q(S_q) = T_q(G \cdot q)^\perp$. A slice satisfying this condition is called an *affine* slice. It can be identified with $T_q(G \cdot q)^\perp$. The dimension of an affine slice is always $\dim(Q) - \dim(G) + \dim(G_q)$.

Lemma A.1. *Any non-maximal point $\tilde{q} \in Q - Q^*$ belongs to the closure (in Q) of some slice $S_q \in Q^*$, such that the closure (in Q) of S_q intersects any orbit at most once.*

Proof. Consider the slice $S_{\tilde{q}}$ at $\tilde{q} \in Q - Q^*$. The slice $S_{\tilde{q}}$ is a $G_{\tilde{q}}$-space, where $\tilde{q}$ is a fixed point of the action. By property (c) above, any slice defined in $S_{\tilde{q}}$ is also a slice in Q, and any orbit in the neighbourhood of $\tilde{q}$ has a counterpart in $S_{\tilde{q}}$. Further, by prescription (A.1), taking closures of subsets of $S_{\tilde{q}}$ in Q is the same as taking closures in $S_{\tilde{q}}$, since $(T_{\tilde{q}}(G\cdot\tilde{q}))^{\perp}$ is a closed subspace of $T_{\tilde{q}}Q$. Hence we can discuss everything in this (Riemannian) submanifold $S_{\tilde{q}}$.

Let us take any point of maximal type $q \in S^*_{\tilde{q}} \subset_{\text{dense}} S_{\tilde{q}}$ sufficiently close to the point $\tilde{q}$. There exists $\exp(tX_q)$, where $0 \le t \le \epsilon$ and $|X_q| = 1$, a geodesic, connecting $\tilde{q}$ and q [1]. An isometric action maps geodesics to geodesics, as well as preserves local distances. Thus together with the smoothness of the group action and the (local) uniqueness of geodesics, this implies that the segment $\exp(tX_q)$ with $0 \le t < \epsilon$ must lie inside $S^*_{\tilde{q}}$; the isotropy group of any points on the segment must be the same as the isotropy group G_q of q; no two points on the segment belong to the same orbit (otherwise isometricity is violated). In fact this geodesic segment is contained in an affine slice, since by (A.1), it is evident that every small enough neighbourhood of any point on this geodesic segment is contained in some affine slice, on which G_q acts trivially. Hence we can choose a sequence of points on the geodesic, converging to $\tilde{q}$, such that the union S of the affine slice at each point in the sequence covers the geodesic segment. This union S satisfies the definition of a slice.

Suppose the closure of S intersects some orbit $\mathcal{O}$ at more than one point. Let us denote these intersecting points by q_g. There can only be countably many such points, by (A.1). Consider the neighbourhood $N_{q_g} \subset S$ of q_g, defined by $N_{q_g} = S_{q_g} \cap S$, where S_{q_g} is some small slice at q_g. By property (a), the set N_{q_g} contains all sequences in S that converge to q_g. $N_{q_g} \cap N_{q_{g'}} = \emptyset$ since $S_{q_g} \cap S_{q_{g'}} = \emptyset$. Hence we can leave out all but one N_{q_g} in our choice of S so that the closure of S intersects the orbit $\mathcal{O}$ only once. We can repeat the same procedure to any other orbits that intersect the closure of S more than once, without disturbing $\tilde{q}$ being in the closure of S, as long as we do not discard any points on the geodesic $\exp(tX_q)$, $0 \le t \le \epsilon$, defined above. By this procedure, we will eventually be left with a choice of S_q that satisfies the requirement. It takes only countably many steps, by property (c) and (A.1). □

Proposition A.2. *On Q^*_0, there is a map τ, which is locally a diffeomorphism (i.e. $\exists$ a neighbourhood around any point, on which τ is diffeomorphic) from Q^*_0 into Q^*. It can be locally Whitney-smoothly extended to Q_0.*

Proof. We first note that there is a local diffeomorphism that maps Q^*_0 into Q^* [9]. Take $q_i \in Q^*$. A small enough slice S_{q_i} can be identified with a suitable neighbourhood $N_{[q_i]} \subset Q^*_0$ of $[q_i] \in Q^*_0$. Let the identification be τ_i so that $\tau_i([q_i]) = q_i$. All points in $\tau_i(N_{[q_i]}) \subseteq S_{q_i}$ have isotropy group G_{q_i} by property (b). Cover Q^*_0 with sets like $N_{[q_i]}$, we can then define a global map τ that is locally defined as above. For consistency of the definition, we notice that the expression of the affine slice implies that in each intersection $N_{[q_i]} \cap N_{[q_{i'}]}$, the transition from diffeomorphism τ_i to $\tau_{i'}$ (i.e. from a slice to another slice on the same orbit) is done via a smooth group action. In general, τ cannot be made into a global diffeomorphism. (In short, Q^* is a fibre bundle with base Q^*_0, and typical fibre G/H, where H is any isotropy group of maximal type.)

Next, we extend the definition of τ so as to include the entire reduced space Q_0. Find a small subset $N^*_{[\tilde{q}]} \subset Q^*_0$ such that its closure $N_{[\tilde{q}]}$ in Q_0 contains $[\tilde{q}] \in Q_0 - Q^*_0$. It is important to stress here that $N_{[\tilde{q}]}$ may not contain all sequences that converge to the point $[\tilde{q}]$, i.e. it is not necessarily a neighbourhood of $[\tilde{q}]$ in Q_0.[5]

We define a local diffeomorphism $\tau_{\tilde{q}}$ for $N^*_{[\tilde{q}]}$ such that $N^*_{[\tilde{q}]}$ is identified with a slice $S_q \subset Q^*$. By Lemma A.1, it is possible to choose S_q so that the closure of S_q contains one and only one point $\tilde{q}$ in the orbit $[\tilde{q}]$. Further, the closure of the slice S_q can be made to intersect any orbits at most once. We extend the map $\tau_{[\tilde{q}]}$ by continuation to the non-maximal points in the closure $N_{[\tilde{q}]}$ of $N^*_{[\tilde{q}]}$,

$$[\tilde{q'}] = \lim_{n\to\infty}[q_n], \qquad [q_n] \in N^*_{[\tilde{q}]}, \qquad \tau_{\tilde{q}}([\tilde{q'}]) = \lim_{n\to\infty}\tau_{\tilde{q}}([q_n]). \tag{A.2}$$

For this to be well-defined, we need to show that first, the required limit exists, and second the limit does not depend on the sequences chosen.

By property (c) of slice, $N_{[\tilde{q}]}$ can be identified with some $S_{[\tilde{q}]}/G_{[\tilde{q}]}$, and $S_{[\tilde{q}]} \supset S_q = \tau_{\tilde{q}}(N^*_{[\tilde{q}]})$ cf. proof of Lemma A.1. For any $[\tilde{q'}] = \lim_{n\to\infty}[q_n]$, the sequence $\tau_{\tilde{q}}([q_n])$ must at least converge to the orbit of $\tilde{q'}$ in $S_{\tilde{q}}$, by definition of the quotient topology. This means that the closure of the slice $S_q = \tau_{[\tilde{q}]}(N^*_{[\tilde{q}]})$ must have non-empty intersection with any neighbourhood of the orbit of $\tilde{q'}$. But, any orbit in $S_{\tilde{q}}$ is compact since $G_{[\tilde{q}]}$ is compact, cf. property (b) and (c). Hence, the closure of $S_q = \tau_{[\tilde{q}]}(N^*_{[\tilde{q}]})$ must intersect the orbit of $\tilde{q'}$ at some points; further, since the closure of the slice S_q intersects any orbit at most once, the limit is unique; thus (A.2) is well-defined.

The extended map $\tau_{\tilde{q}}$ is Whitney-smooth (cf. Section 2.1). This way we have constructed a map τ which is locally Whitney-smooth for the entire Q_0. □

The map τ, being locally Whitney-smooth, pushes tangent spaces on Q_0 onto corresponding tangent spaces on Q. For any $[q] \in Q_0$, τ maps either into or onto $(T_q(G\cdot\tau(q)))^{\perp}$, as in Section 2.1.

The map $T\tau$ provided us a metric, denoted $\tilde{g}$, on Q_0 by

$$\tilde{g}(X_{[q]}, Y_{[q]}) = g(T\tau(X_{[q]}), T\tau(Y_{[q]})). \tag{A.3}$$

We deliberately dropped the index labelling the open sets on which local diffeomorphisms are defined. We see that the G-invariance of the metric g enables us to globally define a metric $\tilde{g}$ on Q_0 via what are merely local diffeomorphisms: Suppose we have $[q'] \in (N_1 \cap N_2) \subset Q_0$, where N_1 and N_2 are two patches on Q_0 where we defined our local diffeomorphism. Let $\tau_i(q') = q'_i$, $i = 1, 2$, and $q'_2 = g_{21}\cdot q'_1$. Then for $X \in T_{[q']}Q_0$, $T\tau_2(X)$ and $T\tau_1(X)$ are related by $\lim_{t\to 0}(1/t)(g_{21}\cdot(q'_1 + T\tau_1(X)t) - g_{21}\cdot q'_1) = T\Phi|_{(g_{21},q'_1)}(X)$, where '$\Phi$' is the group action. But the metric g is invariant under $T\Phi$, so the metric defined above is independent of the choice of local diffeomorphisms.

As in Section 2, let P be the homogeneous space associated with the maximal orbits. We can form the space $M = Q_0 \times P$, carrying the action of G based on the left translation of G on P (recall that P is diffeomorphic to some G/T, where T is an isotropy group of Q

[5] If Q_0 is a cone, and $[\tilde{q}]$ is the tip, the neighbourhood of $[\tilde{q}]$ is simply a small cone with the same tip; but the set $N_{[\tilde{q}]}$ may only be half of the small cone, but containing the tip.

of maximal type): $g \cdot ([q], p) = ([q], L_g \cdot p)$, $g \in G$, $[q] \in Q_0$, $p \in P$. The (conical) tangent space on M by definition is the direct sum of the (conical) tangent space on Q_0 and the tangent space on P.

Proposition A.3. *There is a locally (Whitney-) smooth equivariant map π from $M = Q_0 \times P$ onto Q.*

Proof. For any $[q_k] \in N_{[q_k]} \subset Q_0^*$, $\tau_k([q_k]) \times P$, where $N_{[q_k]}$ is the neighbourhood set, find a diffeomorphism κ_k that maps P to $G/G_{\tau_k([q_k])}$. Then we can identify $(\tau_k([q_k]), \kappa_k(P))$ with $G \cdot q_k$. The equivariant identification π_k' is given by $g_H \cdot \tau_k([q_k])$, $g_H \in \kappa_k(P)$ [B]. Note that $G_{\tau_k([q_k])}$ acts trivially on $\tau_k(N_{[q_k]})$, which by definition, is some affine slice $S_{\tau_k([q_k])}$. The map $\pi_k' \circ (\tau_k, \kappa_k)$ is therefore well-defined for all points in $N_{[q_k]}$, and is therefore a diffeomorphism on $\tau_k(N_{[q_k]}) \times \kappa_k(P)$.

Consider the subset $N_{\tilde{q}} \subset Q_0$ containing non-maximal points, as constructed in the proof of Proposition A.2. Let $N_{\tilde{q}}^* \subset Q_0^*$ be its dense subset. We first have a diffeomorphism $\pi' \circ (\tau_{\tilde{q}}, \kappa_{\tilde{q}})$ defined on $N_{\tilde{q}}^* \times P$ as above, where $\tau_{\tilde{q}}$ is the diffeomorphism defined on $N_{\tilde{q}}^*$ and on $N_{\tilde{q}}$ by extension (cf. Proposition A.2), and $\kappa_{\tilde{q}}$ identifies P with G/H, where H is the (common) isotropy group of points in $\tau_{\tilde{q}}(N_{\tilde{q}}^*)$. Then $\pi' \circ (\tau_{\tilde{q}}, \kappa_{\tilde{q}})$ is smoothly extended to all of $N_{\tilde{q}}$ when we extend $\tau_{\tilde{q}}$ to the entire neighbourhood $N_{\tilde{q}}$. We see that the map is well defined for all points of the neighbourhood, since the isotropy group of all points in $\tau_{\tilde{q}}(N_{\tilde{q}})$ is or contains H, and $\tau_{\tilde{q}}(N_{\tilde{q}})$ intersects any orbit at most once.

Let us cover M by subspace of the form $N_k \times P$, where N_k is a subset of Q_0, like the one above. We can define a smooth projection π_k for each of the subspace, where π_k is the composition $\pi_k' \circ (\tau_k, \kappa_k)$, π_k' and κ_k are as defined above. The projection π_k is Whitney-smooth on $N_{[q_k]} \times P$.

Let us denote the overall projection, which is locally Whitney-smooth, as π. It is evidently equivariant (since all π_k' is). □

For any $p \in P$, the space $Q_0 \times \{p\}$, denoted as (Q_0, p), satisfies the following conditions: for any $([q], p) \in (Q_0, p)$,
(1) $G \cdot ([q], p) \cap (Q_0, p) = \{([q], p)\}$;
(2) $T^C_{([q],p)} M = T^C_{([q],p)}((Q_0, p)) \oplus T_{([q],p)}(G \cdot ([g], p))$.
These conditions would define 'transverse' sub-manifolds [14], except that our (Q_0, p) may contain singularities. Nevertheless, we shall call (Q_0, p), which can be identified with Q_0, transverse. Note that (Q_0^*, p) is indeed transverse in M^* in the conventional sense.

The projection π induces a metric on M via the pull-back of π. In other words, the metric is defined as $g \circ T\pi$, where g is the original metric on Q. Again, although $T\pi$ is only locally defined, $g \circ T\pi$ is independent of the choice of local diffeomorphisms (the same as the argument for $\tilde{g}$). Thus the metric is globally defined. But again, it is degenerate on $M - M^*$ (cf. (2.11)). Let us also denote the induced metric by g. The decomposition of this metric and the factorization of the measure $\mathrm{d}m$ follow as before.

The above discussion implies that the Hilbert space $L^2(Q, \mathrm{d}q)$ is naturally unitarily equivalent to the Hilbert space $L^2(M, \mathrm{d}m) = L^2(Q_0, \rho([q])\,\mathrm{d}[q]) \otimes L^2(P, \mathrm{d}p)$. It can be shown as in Section 2, by taking the set $\tilde{C}_c^\infty(Q^*)$ of compactly defined locally smooth functions on Q^*, instead of globally smooth functions. Under the pull-back π^* of the (locally Whitney-smooth) projection π, the image of $\tilde{C}_c^\infty(Q^*)$ is denoted as $\pi^*\tilde{C}_c^\infty(Q^*)$. This map then extends to an isomorphism bewteen the two Hilbert spaces.

Under the unitary map π^*, the unconstrained Laplacian Δ is transformed to Δ_* defined by (2.14). Recall that Δ can be defined on $C_c^\infty(Q)$, and such a Δ is e.s-a. Since the set of locally smooth functions $\tilde{C}_c^\infty(Q)$ contains $C_c^\infty(Q)$, and it is contained in the s-a. domain of Δ, we can use $\tilde{C}_c^\infty(Q)$ as an alternative e.s-a. domain of Δ. From Eq. (2.14), it is easy to see that Δ_* is also e.s-a. when defined on $\pi^*\tilde{C}_c^\infty(Q)$.

Comparing this to the construction in Section 2, there is only one more complication. Namely, the image of any function $f \in \tilde{C}_c^\infty(Q)$ under π^* depends on the arbitrary choice made in constructing each local diffeomorphism from Q_0 into Q. Does that make Δ_* more or less arbitrarily defined? Recall that the original Laplacian Δ is a G-invariant operator, i.e. $\Delta(U(g)f) = U(g)\Delta(f)$, where $U(g)$ is the representation of G on $L^2(Q, \mathrm{d}q)$ (defined as $U(g)f = f \circ g^{-1}$, cf. Section 1.2). But, in the notation of the argument following (A.3) and Proposition A.3, given $[q] \in (N_1 \cap N_2) \subset Q_0$, $\tau_1([q]) = q_1$, $\tau_2([q]) = g_{21} \cdot q_1$, we have

$$\pi_i^* f([q], p) = f \circ \pi_i([q], p) = f(\kappa_i(p) \cdot \tau_i[q]),$$

where $i = 1, 2$. Hence $\pi_1^* f([q], p)$ and $\pi_2^* f([q], p)$ are related by the action of some $U(g)$. Further, since π is equivariant, π^* commutes with the unitary operation of $U(g)$. Let Δ_*^1 and Δ_*^2 be the Laplacians defined with respect to the two choices π_1^* and π_2^*, respectively. It is easy to check that $\Delta_*^2(\pi_1^* f) = \Delta_*^1(\pi_1^* f)$. Thus the definition of Δ_* is independent of the local choice of π^*.

Such a Hamiltonian Δ_* is precisely the one defined by the metric induced on M. By defining the Laplacian on $\pi^*\tilde{C}^\infty(Q)$, the action of the Laplacian remains (locally) meaningful even on points in $M - M^*$ where the metric is degenerate.

For the smooth invariant functions on Q, $C^\infty(Q)^G$, the image $\pi^*C^\infty(Q)^G$ is independent of the choice of local diffeomorphism, so that the important function space $C^\infty(Q_0)_{(0)} = \pi^*\tilde{C}^\infty(Q)^G|_{Q_0} = \pi^*C^\infty(Q)^G|_{Q_0}$ (defined in Sections 2.1) is *independent* of the choice of the local diffeomorphism.

We have now completed the construction of all the essential objects. The rest of the analysis in Sections 2.4 and 2.5 follows straightforwardly.

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